Improved Lower Bounds on the Randomized Complexity of Graph Properties*

Amit Chakrabarti
Dartmouth College
ac@cs.dartmouth.edu †

Subhash Khot
Georgia Institute of Technology
khot@cc.gatech.edu ‡

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Abstract

We prove a lower bound of $\Omega(n^{4/3}\log^{1/3} n)$ on the randomized decision tree complexity of any nontrivial monotone n-vertex graph property, and of any nontrivial monotone bipartite graph property with bipartitions of size n. This improves the previous best bound of $\Omega(n^{4/3})$ due to Hajnal [Haj91]. Our proof works by improving a graph packing lemma used in earlier work, and this improvement in turn stems from a novel probabilistic analysis. Graph packing being a well-studied subject in its own right, our improved packing lemma and the probabilistic technique used to prove it may be of independent interest.

Keywords: Decision trees, graph properties, complexity, randomized algorithms, graph packing, probabilistic method.

AMS Subject Classifications: 68Q17 (Computational difficulty of problems), 68Q25 (Analysis of algorithms and problem complexity), 68R10 (Graph theory).

1 Introduction

Consider the problem of deciding whether or not a given input graph G has a certain (isomorphism invariant) property P. The graph is given by an oracle which answers queries of the form "Is (x, y) an edge of G?" A *decision tree algorithm* for P makes a sequence of such queries to the oracle, where each query may depend upon the information obtained from the previous ones, until sufficient information about G has been obtained to decide whether or not P holds for G, whereupon it either accepts or rejects. Let $\mathscr{A}_P$ denote the set of decision tree algorithms for P and, for $A \in \mathscr{A}_P$, let $\mathrm{cost}(A, G)$ denote the number of queries that A asks on input G. The quantity $\mathcal{C}(P) = \min_A \max_G \mathrm{cost}(A, G)$ is called the *deterministic decision tree complexity*, or simply the deterministic complexity, of P.

A *randomized decision tree algorithm* for P is a probability distribution μ over $\mathscr{A}_P$, and its cost (on input G) is the expectation of $\mathrm{cost}(A, G)$ with A drawn from μ:

$$\mathrm{cost}^R(\mu, G) \quad = \quad \sum_{A\in\mathscr{A}_P} \Pr_\mu[A]\,\mathrm{cost}(A, G) \quad .$$

The *randomized decision tree complexity*, or simply the randomized complexity, of P is defined to be

$$\mathcal{C}^R(P) \quad = \quad \min_\mu \max_G \mathrm{cost}^R(\mu, G) \quad .$$

An n-vertex graph property is said to be *nontrivial* if there is at least one n-vertex graph which has the property and at least one which does not. It is said to be *monotone* if addition of edges does not destroy the property. Let $\mathscr{P}_n$

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†Computer Science Department, Dartmouth College, Hanover, NH 03755.

‡College of Computing, Georgia Institute of Technology, 801 Atlantic Drive, Atlanta, GA 30332.

denote the set of all nontrivial monotone n-vertex graph properties. Since any property can be decided by the brute force strategy of querying all possible vertex pairs, the following upper bounds clearly hold:

$$\forall\ P \in \mathscr{P}_n : \quad \mathcal{C}^R(P) \ \le\ \mathcal{C}(P) \ \le\ \binom{n}{2} \ =\ O(n^2)\ . \tag{1}$$

The main result of this paper is the following lower bound.

Theorem 1.1 (Main Theorem) *Any property $P \in \mathscr{P}_n$ satisfies $\mathcal{C}^R(P) = \Omega(n^{4/3}\log^{1/3} n)$.*

Our proof of this lower bound relies on an important theorem from the pioneering work of Yao [Yao87], as well as on a framework developed by Hajnal [Haj91]. In this framework we associate with a graph property a special pair of graphs which cannot be "packed" together. We then argue, using Yao's results, that if the property has low randomized complexity, then certain degree upper bounds hold for these special graphs. Finally, we use these degree bounds to prove that the special graphs *can* be packed, thereby arriving at a contradiction.

The notion of graph packing, which we shall formally define later, is a well-studied subject in its own right [Bol78, Chapter 8]. A packing lemma (Lemma 3.10) we establish in this paper is therefore of independent interest since it improves a packing theorem due to Hajnal and Szegedy [HS92].

The rest of the paper is organized as follows. We quickly survey some related work in Section 2. In Section 3, we introduce some preliminary notions, describe the framework alluded to above and outline a proof of Theorem 1.1 using this framework. In Section 4, we prove a couple of technical lemmas to analyze the behavior of hypergraphs under random vertex deletions. These lemmas are then used in Section 5 to prove our improved packing lemma. The brief Section 6 wraps up the proof of Theorem 1.1.

2 Related Work

The decision tree complexity of Boolean functions is one of the core areas of complexity theory, and one that has been studied since the 1970s. Graph properties, being particularly natural and interesting Boolean functions, have been an important focus of this study. Yet, two very basic conjectures about the decision tree complexity of graph properties have frustrated about 30 years of attack.

We have already noted the trivial upper bounds given by (1). A classic result of Rivest and Vuillemin [RV76] gives a lower bound $\mathcal{C}(P) = \Omega(n^2)$, for all $P \in \mathscr{P}_n$, which settles the deterministic complexity of monotone graph properties up to a constant. However, in the world of deterministic complexity, a far more interesting conjecture is that any $P \in \mathscr{P}_n$ has $\mathcal{C}(P) = \binom{n}{2}$ exactly. Remarkably, this bold conjecture, attributed to Karp, remains open to this day. Some important special cases of the conjecture have been settled; for example, a seminal result of Kahn, Saks and Sturtevant [KSS84] proves the truth of the conjecture when n is a prime power, a result of Yao [Yao88] does the same for bipartite graph properties, and a recent result of Chakrabarti, Khot and Shi [CKS02] settles the conjecture for properties closed under graph minors. See [CKS02] and the references therein for more details on this line of work.

Returning to randomized complexity (the focus of this paper), the first nonlinear lower bound on $\mathcal{C}^R(P)$, for general $P \in \mathscr{P}_n$, was an $\Omega(n\log^{1/12} n)$ bound proven by Yao [Yao87]. This was subsequently improved by King [Kin88] to $\Omega(n^{5/4})$ and later by Hajnal [Haj91] to $\Omega(n^{4/3})$. Our main theorem clearly improves on all of the above results.

There are two other significant papers in the area with results incomparable to ours. Gröger [Grö92] established lower bounds stronger than we do for certain special classes of graph properties, including Hamiltonicity, k-colorability and subgraph containment. He also established an important link between our problem and that of proving lower bounds on the randomized complexity of monotone *bipartite* graph properties, stated as Fact 3.2 in this paper. More recently, Friedgut, Kahn and Wigderson [FKW02] showed an intriguing connection between the complexity $\mathcal{C}^R(P)$ of a property $P \in \mathscr{P}_n$ and its threshold probability $\theta(P)$, defined as the infimum of all p for which the random graph $G(n, p)$ has property P with probability at least $\frac{1}{2}$. Their result $\mathcal{C}^R(P) = \Omega(n^2/\max\{\theta(P)n, \log n\})$ is incomparable to ours because the bound can be as bad as $\Omega(n)$, though for some natural properties it is as high as the near-optimal $\Omega(n^2/\log n)$.

No property in $\mathscr{P}_n$ is known to have randomized complexity below $n^2/4$. Closing this gap between our lower bound and this upper bound is one of the most important open problems concerning decision tree complexity in general, and graph properties in particular.

We remark that monotonicity is crucial for our result as well as all of those cited above. There are examples of nontrivial non-monotone graph properties with $\mathcal{C}(P) = O(n)$, the most famous being the "scorpion" property; see [Bol78] or [DK00, Exercise 5.15] for details.

3 Preliminaries and Proof Outline

We now describe several concepts and earlier results (stated here as "facts") that are used in the proof of the main theorem. We also give an outline of that proof.

The first step is to change the objects of study from graph properties to *bipartite* graph properties, which we now define.

Definition 3.1 (Bipartite graphs, degrees, complements) *An (m,n)-bipartite graph G is a graph whose vertices are partitioned into two independent sets, denoted $V_L(G)$ and $V_R(G)$ respectively, of sizes m and n respectively. The edge set of G is denoted $E(G)$. For such a graph we define*

$$\Delta_L(G) = \max_{v \in V_L(G)} \deg_G(v) \ , \qquad \delta_L(G) = \frac{1}{|V_L(G)|} \sum_{v \in V_L(G)} \deg_G(v) \ = \frac{|E(G)|}{|V_L(G)|} \ .$$

$\Delta_R(G)$ and $\delta_R(G)$ are defined similarly. When $|V_L(G)| = |V_R(G)|$ we define $\delta(G) = \delta_L(G) = \delta_R(G)$. We define the bipartite complement $\bar{G}$ of G to be the (m,n)-bipartite graph with the same bipartition and with edge set $V_L(G) \times V_R(G) - E(G)$.

Let $\mathscr{P}_{n,n}$ denote the set of all nontrivial monotone properties of (n,n)-bipartite graphs. The randomized complexity $\mathcal{C}^R(P)$ of a property $P \in \mathscr{P}_{n,n}$ is defined analogously to that of a property in $\mathscr{P}_n$. A result of Gröger [Grö92, Theorem 3.5] provides a vital link between the two notions.

Fact 3.2 ([Grö92]) *Let $f(n)$ be a function satisfying $f(n) = O(n^{3/2})$ and suppose any $P \in \mathscr{P}_{n,n}$ satisfies $\mathcal{C}^R(P) = \Omega(f(n))$. Then any $Q \in \mathscr{P}_n$ satisfies $\mathcal{C}^R(Q) = \Omega(f(n))$.*

Since our main theorem claims a lower bound of only $\Omega(n^{4/3} \log^{1/3} n)$, we may therefore safely concentrate on monotone bipartite graph properties alone. We need some further definitions.

Definition 3.3 (Dual) *The* dual *of a property $P \in \mathscr{P}_{n,n}$ is defined to be the property $P^* \in \mathscr{P}_{n,n}$ such that a graph G satisfies P^* iff $\bar{G}$ does not satisfy P.*

Definition 3.4 (Minterms) *An (n,n)-bipartite graph G is called a* minterm *of $P \in \mathscr{P}_{n,n}$ if G satisfies P but removing any edge from G yields a graph which does not.*

Definition 3.5 (Sparseness) *The bipartite graph G is said to be L-*sparse *if $V_L(G)$ contains at least $\frac{1}{2}|V_L(G)|$ isolated vertices, i.e., vertices of degree* 0*. The notion of R-*sparseness *is defined analogously.*

Sparse minterms play a very crucial role in our argument. Suppose we associate with G an n-tuple $(d_1, d_2, \ldots, d_n)$ with $d_1 \geq \ldots \geq d_n$, where the d_i are the degrees of the vertices in $V_L(G)$; we then say that G is an *L-first minterm* of P if it is a minterm and its associated n-tuple is lexicographically smallest amongst all minterms. We say that G is an *L-first sparse minterm* of P if it is a minterm, is L-sparse, and its associated n-tuple is lexicographically smallest amongst all L-sparse minterms. We define *R-first minterms* and *R-first sparse minterms* analogously.

Lemma 3.6 *For all $P \in \mathscr{P}_{n,n}$, with n even, either P or P^* has an R-sparse minterm.*

Proof: Let G be the graph obtained by adding $n/2$ isolated vertices to the complete bipartite graph $K_{n,n/2}$. By appropriate labeling, G can be made an R-sparse (n,n)-bipartite graph. Note that $\bar{G}$ is isomorphic to G. Therefore G must satisfy either P or P^* and the lemma follows. ∎

It is easy to see that any decision tree algorithm for P can be converted into one for P^*; this gives $\mathcal{C}^R(P) = \mathcal{C}^R(P^*)$. Therefore, from now on we shall assume, without loss of generality, that P has an R-sparse minterm. We now state the key result of Yao [Yao87] and an extension of the result due to Hajnal [Haj91].

Fact 3.7 ([Yao87, Haj91]) *For $P \in \mathscr{P}_{n,n}$, the following hold*
(1) *If G is a minterm of P then $\mathcal{C}^R(P) = \Omega(|E(G)|)$.*
(2) *If G is either an L-first minterm or an L-first sparse minterm, then*

$$\mathcal{C}^R(P) = \Omega(n\Delta_L(G)/\delta_L(G)) \ ,$$

and a similar statement holds for R-first minterms and R-first sparse minterms. ∎

Let us say that graphs G and H *can be packed* if there is a way to identify their vertices without identifying any edge of G with an edge of H. Such an identification, when it exists, shall be called a *packing* of G and H. To see the relevance of this concept, consider the case when G and H are minterms of P and P^* respectively, for some property $P \in \mathscr{P}_{n,n}$. To say that G and H can be packed is equivalent to saying that G is isomorphic to a subgraph of $\bar{H}$. Since G is a minterm of P and P is monotone, this means $\bar{H}$ satisfies P. However, H is a minterm of the dual P^*, so $\bar{H}$ does not satisfy P. This contradiction shows that G and H cannot be packed.

These ideas are formalized in the next definition[1] and the following fact.

Definition 3.8 (Packing) *Let G and H be (m, n)-bipartite graphs. A* packing *of G and H is a pair of bijections $\sigma_L : V_L(G) \to V_L(H)$ and $\sigma_R : V_R(G) \to V_R(H)$ such that, for all $(x, y) \in V_L(G) \times V_R(G)$, either $(x, y) \notin E(G)$ or $(\sigma_L(x), \sigma_R(y)) \notin E(H)$. We say that G and H* can be packed *if there exists such a packing.*

Fact 3.9 ([Yao87]) *For $P \in \mathscr{P}_{n,n}$, let G be a minterm of P and H be a minterm of P^*. Then G and H cannot be packed.* ∎

Finally, we outline the proof of Theorem 1.1. Let $P \in \mathscr{P}_{n,n}$ and let $q = q(n)$ be a parameter to be fixed later. We wish to prove that $\mathcal{C}^R(P) = \Omega(nq)$. Suppose this is not the case. Let G be an R-first sparse minterm of P and H be an L-first minterm of P^*. By part (1) of Fact 3.7, the following conditions hold:

$$\delta(G) \le q \ , \qquad \delta(H) \le q \ .$$

Using these in part (2) of Fact 3.7 gives us the following additional conditions:

$$\Delta_R(G) \le q^2 \ , \qquad \Delta_L(H) \le q^2 \ .$$

We would like to show that for an appropriate choice of q, these conditions imply that G and H can be packed, which would contradict Fact 3.9.

The above framework is the same as that used by Hajnal [Haj91]. Our improvement is in the parameters of the packing lemma. Our improved lemma says:

Lemma 3.10 (Packing Lemma) *Set $q = (\alpha n \ln n)^{1/3}$. Let G and H be (n, n)-bipartite graphs with $\delta(G) \le q$, $\delta(H) \le q$, $\Delta_R(G) \le q^2$ and $\Delta_L(H) \le q^2$. Furthermore, suppose G is R-sparse. Then, if α is a small enough constant, G and H can be packed.*

The above result is stronger than all earlier bipartite graph packing results in the following sense. All earlier results, including the theorem of Hajnal and Szegedy [HS92] that was used in Hajnal's work on graph properties [Haj91], require conditions at least as strong as $\Delta_R(G)\delta_L(H) \le O(n)$ and $\delta_R(G)\Delta_L(H) \le O(n)$. However, we allow these products to exceed n and go up to $\Theta(n \log n)$. This makes Lemma 3.10 interesting on its own.

As noted above, this lemma eventually implies a lower bound of $\Omega(nq) = \Omega(n^{4/3} \log^{1/3} n)$ on the randomized complexity of graph properties, as claimed by the Main Theorem. Thus, our new goal is to prove Lemma 3.10. Our proof will use a probabilistic technique that requires an understanding of the following question: how many hyperedges does a hypergraph lose when a random subset of its vertices is deleted? The next section provides two answers to this question, both of which will be useful later.

4 Random Vertex Deletion in Hypergraphs

Definition 4.1 (Hypergraphs, degrees, squarishness) *A hypergraph $\mathcal{H}$ consists of a finite set $V(\mathcal{H})$ of* vertices *and a collection $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$, called* hyperedges*; $E(\mathcal{H})$ may contain multiple copies of the same subset of $V(\mathcal{H})$ and may contain the empty set. For a vertex v, its* degree $\deg_{\mathcal{H}}(v)$ *is defined to be the number of hyperedges (counting multiplicities) that contain v. The hypergraph $\mathcal{H}$ is said to be* squarish *if $\frac{1}{2}|V(\mathcal{H})| \le |E(\mathcal{H})| \le |V(\mathcal{H})|$.*

For readers familiar with hypergraphs in other contexts, we note that our hypergraphs, as defined above, are not necessarily uniform or simple.

[1] We have defined the notion of packing only for bipartite graphs here because that is all we need. In the literature, packing has been studied both for general graphs as well as bipartite graphs.

Definition 4.2 (Vertex deletion) *Let $\mathcal{H}$ be a hypergraph and $T \subseteq V(\mathcal{H})$. The hypergraph obtained by deleting T from $\mathcal{H}$, denoted $\mathcal{H} \setminus T$, is defined by*

$$\begin{aligned} V(\mathcal{H} \setminus T) &= V(\mathcal{H}) \setminus T \ , \\ E(\mathcal{H} \setminus T) &= \{A \in E(\mathcal{H}) : A \subseteq V(\mathcal{H}) \setminus T\} \ . \end{aligned}$$

Definition 4.3 (Favorable hypergraphs) *A hypergraph on a nonempty vertex set is said to be $(r, s, \bar{s})$-favorable if*

- *every hyperedge has size at most r,*
- *every vertex has degree at most s, and*
- *the average degree of a vertex is at most $\bar{s}$.*

Now consider deleting, from an $(r, s, \bar{s})$-favorable hypergraph $\mathcal{H}$, a randomly chosen subset of t of its vertices. We would like to provide sufficient conditions that ensure that, with high probability, a large number of hyperedges will survive this deletion. It is easy to see that suitable upper bounds on r and t ensure that the *expected* number of surviving hyperedges is large. But the *high probability* results we seek need more conditions, such as upper bounds on s and $\bar{s}$, in order to avoid situations where the hyperedges are all concentrated on just a few vertices. We now state two such results precisely.

Lemma 4.4 *Let $\mathcal{H}$ be an $(r, s, \bar{s})$-favorable hypergraph on n vertices, where n is sufficiently large. Let T be a random subset of $V(\mathcal{H})$ of size t, chosen uniformly from amongst all such subsets. Suppose $s \le n^{1-3\varepsilon}$ and $t\bar{s} \le n^{1-3\varepsilon}$, for some constant $\varepsilon > 0$. Then*

$$\Pr\Big[|E(\mathcal{H} \setminus T)| < |E(\mathcal{H})| - n^{1-2\varepsilon}\Big] \quad \le \quad \frac{1}{n^2} \ .$$

Proof: Suppose $V(\mathcal{H}) = \{1, 2, \ldots, n\}$. Define the Boolean random variable X_i by $X_i = 1$ iff $i \in T$. Let d_i be the degree of vertex i. Since the deletion of vertex i can kill at most d_i hyperedges not already killed by other vertex deletions,

$$|E(\mathcal{H} \setminus T)| \quad \ge \quad |E(\mathcal{H})| - \sum_{i=1}^{n} d_i X_i \ .$$

Note that the expectation $\mathrm{E}\left[\sum_{i=1}^{n} d_i X_i\right] = \sum_{i=1}^{n} d_i t/n = t\bar{s}$. Note also that the random variables $d_i X_i / s$ are distributed in the interval $[0, 1]$. Thus, for any $\lambda > 0$, the Chernoff-Hoeffding bound for the hypergeometric distribution [Hoe63, Theorems 1 and 4] gives us

$$\Pr\left[\sum_{i=1}^{n} d_i X_i > (1+\lambda)t\bar{s}\right] \ = \ \Pr\left[\sum_{i=1}^{n} \frac{d_i X_i}{s} > \frac{(1+\lambda)t\bar{s}}{s}\right] \ \le \ \left(\frac{e^{\lambda}}{(1+\lambda)^{1+\lambda}}\right)^{t\bar{s}/s} \ .$$

Set $1+\lambda = n^{1-2\varepsilon}/(t\bar{s})$. The condition $t\bar{s} \le n^{1-3\varepsilon}$ ensures that this λ is large enough to imply $e^{\lambda}/(1+\lambda)^{1+\lambda} \le e^{-(1+\lambda)}$. Therefore

$$\Pr\left[\sum_{i=1}^{n} d_i X_i > n^{1-2\varepsilon}\right] \ \le \ \exp\left(-\frac{(1+\lambda)t\bar{s}}{s}\right) \ = \ \exp\left(-\frac{n^{1-2\varepsilon}}{s}\right) \ .$$

The condition $s \le n^{1-3\varepsilon}$ ensures that this is at most $\exp(-n^{\varepsilon}) \le n^{-2}$, for sufficiently large n. ∎

Lemma 4.5 *Let $\mathcal{H}$ be an $(r, s, \bar{s})$-favorable squarish hypergraph on n vertices, where n is sufficiently large. Let T be a random subset of $V(\mathcal{H})$ of size t, chosen uniformly from amongst all such subsets. Suppose $s \le n^{1-2\varepsilon}$, $rt \le \frac{1}{8}\varepsilon n \ln n$ and $ts\bar{s} \le n^{2-3\varepsilon}$, for some constant $\varepsilon > 0$. Then*

$$\Pr\Big[\big|E(\mathcal{H} \setminus T)\big| < n^{1-2\varepsilon}\Big] \quad \le \quad \frac{1}{n^2} \ .$$

The proof of Lemma 4.5 turns out to be surprisingly tricky, and involves a rather delicate tuning of parameters. It also requires a nontrivial tail estimate for a sum of *dependent* random variables, given by the following lemma.

Lemma 4.6 *Let $d_1, \ldots, d_n$ be non-negative integers, $\delta = \frac{1}{n}\sum_{i=1}^{n} d_i$ and $\Delta = \max\{d_1, \ldots, d_n\}$. Suppose $Z_1, \ldots, Z_n$ are independent and identically distributed Boolean random variables with $\Pr[Z_i = 1] = p$. Then, for any choice of functions $\psi_i : \{0,1\}^{i-1} \to [0, d_i]$ and any $a > 0$, we have*

$$\Pr\left[\sum_{i=1}^{n}(p - Z_i)\psi_i(Z_1, \ldots, Z_{i-1}) \;<\; -a\right] \;\leq\; \exp\left(\frac{pn\delta}{\Delta}\left(\frac{a}{pn\delta} - \left(1 + \frac{a}{pn\delta}\right)\ln\left(1 + \frac{a}{pn\delta}\right)\right)\right) \;.$$

Proof: Let $X_k = \sum_{i=k+1}^{n}(p - Z_i)\psi_i(Z_1, \ldots, Z_{i-1})$. Note that $\mathrm{E}[X_0] = 0$; thus, the lemma can be thought of as asserting that X_0 is unlikely to drop far below its expectation. Although X_0 is a sum of dependent random variables, the nature of the dependencies is such that we can mimic large parts of the proof of standard Chernoff bounds (e.g., see [AS00, Theorems A.11 and A.12]).

Let $\theta > 0$ be a parameter to be fixed later. Let us define the function $f_p : \mathbb{R} \to \mathbb{R}$ by $f_p(x) = pe^{(1-p)x} + (1-p)e^{-px}$. We claim that for all k, $0 \leq k \leq n$, and all $\zeta \in \{0,1\}^k$, the following inequality holds.

$$\mathrm{E}\left[e^{-\theta X_k} \mid Z_1 \cdots Z_k = \zeta\right] \;\leq\; \prod_{i=k+1}^{n} f_p(\theta d_i) \;. \tag{2}$$

We prove (2) by reverse induction on k. For $k = n$ it is trivially true: the empty sum defaults to zero and the empty product to unity. For a particular $k < n$, we proceed as follows.

$$\begin{aligned}
\mathrm{E}\left[e^{-\theta X_k} \mid Z_1 \cdots Z_k = \zeta\right] &= \mathrm{E}\left[e^{-\theta(p - Z_{k+1})\psi_{k+1}(Z_1, \ldots, Z_k)} e^{-\theta X_{k+1}} \mid Z_1 \cdots Z_k = \zeta\right] \\
&= pe^{-\theta(p-1)\psi_{k+1}(\zeta)} \cdot \mathrm{E}\left[e^{-\theta X_{k+1}} \mid Z_1 \cdots Z_{k+1} = \zeta \circ 1\right] \\
&\quad + (1-p)e^{-\theta p \psi_{k+1}(\zeta)} \cdot \mathrm{E}\left[e^{-\theta X_{k+1}} \mid Z_1 \cdots Z_{k+1} = \zeta \circ 0\right] \\
&\leq f_p(\theta \psi_{k+1}(\zeta)) \prod_{i=k+2}^{n} f_p(\theta d_i) \qquad (3) \\
&\leq \prod_{i=k+1}^{n} f_p(\theta d_i) \;, \qquad (4)
\end{aligned}$$

where (3) follows from the inductive hypothesis and (4) follows from the fact that f_p is non-decreasing on $\mathbb{R}^+$ because its derivative $f_p'(x) = p(1-p)e^{-px}(e^x - 1)$ is non-negative on $\mathbb{R}^+$. This proves (2) for all k. Setting $k = 0$ gives

$$\mathrm{E}\left[e^{-\theta X_0}\right] \;\leq\; \prod_{i=1}^{n} f_p(\theta d_i) \;=\; \prod_{i=1}^{n} e^{-\theta p d_i} \cdot \prod_{i=1}^{n}\left(pe^{\theta d_i} + 1 - p\right) \;=\; e^{-\theta pn\delta}\prod_{i=1}^{n}\left(pe^{\theta d_i} + 1 - p\right) \;.$$

This latter product, under the constraints $\sum_{i=1}^{n} d_i = n\delta$ and $\max\{d_1, \ldots, d_n\} = \Delta$, is maximized when $n\delta/\Delta$ of the d_i's are equal to Δ and the rest are equal to 0. This can be proved using a straightforward argument that considers varying exactly two of the d_i's at a time; we omit the details. Using this, we obtain

$$\mathrm{E}\left[e^{-\theta X_0}\right] \;\leq\; e^{-\theta pn\delta}\left(pe^{\theta\Delta} + 1 - p\right)^{n\delta/\Delta} \;.$$

An application of Markov's inequality gives

$$\Pr[X_0 < -a] \;=\; \Pr\left[e^{-\theta X_0} > e^{\theta a}\right] \;\leq\; e^{-\theta pn\delta - \theta a}\left(pe^{\theta\Delta} + 1 - p\right)^{n\delta/\Delta} \;.$$

Setting $\theta = (1/\Delta)\ln(1 + a/(pn\delta))$ and using the fact $(1 + a/n\delta)^{n\delta} \leq e^a$, we obtain

$$\begin{aligned}
\Pr[X_0 < -a] &\leq \exp\left(-\frac{a + pn\delta}{\Delta}\ln\left(1 + \frac{a}{pn\delta}\right)\right) \cdot \left(1 + \frac{a}{n\delta}\right)^{n\delta/\Delta} \\
&\leq \exp\left(\frac{pn\delta}{\Delta}\left(\frac{a}{pn\delta} - \left(1 + \frac{a}{pn\delta}\right)\ln\left(1 + \frac{a}{pn\delta}\right)\right)\right) \;,
\end{aligned}$$

which is the desired bound. ∎

We shall now use this tail inequality to prove Lemma 4.5.

Proof of Lemma 4.5: Let the random variable $X(u)$ count the number of hyperedges that remain when we delete from $\mathcal{H}$ a random set of u vertices; the lemma gives an upper bound on $\Pr[X(t) < n^{1-2\varepsilon}]$. Let the random variable $Y(p)$ count the same number when we delete each vertex with probability p, independently. We begin by showing that

$$\Pr[X(u) < a] \quad \leq \quad 2 \cdot \Pr[Y(2u/n) < a] \tag{5}$$

for any real a. The proof is simple and is adapted from a similar observation by Hajnal [Haj91]. For $0 \leq k \leq n$, let $\pi_k = \Pr[X(k) < a]$. Observe that $\pi_0 \leq \pi_1 \leq \cdots \leq \pi_n$. Let $p = 2u/n$, $A = \lfloor \frac{1}{2}np \rfloor = u$ and $B = \lfloor \frac{3}{2}np \rfloor$. We have

$$\Pr[Y(p) < a] \;=\; \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \pi_k \;\geq\; \sum_{k=A}^{B} \binom{n}{k} p^k (1-p)^{n-k} \pi_A \;\geq\; \frac{1}{2}\pi_A \;=\; \frac{1}{2}\pi_u \;,$$

which proves (5). For the rest of the proof, fix $p = 2t/n$. Let us first estimate the expectation $\mathrm{E}[Y(p)]$. Any hyperedge of $\mathcal{H}$ has size at most r, so it survives after the vertex deletions with probability at least $(1-p)^r$. Since $\mathcal{H}$ is squarish, it has at least $\frac{1}{2}n$ hyperedges. Therefore,

$$\begin{aligned}
\mathrm{E}[Y(p)] &\geq \tfrac{1}{2}n(1-p)^r \\
&= \tfrac{1}{2}n\left((1-p)^{1/p}\right)^{2rt/n} \\
&\geq \tfrac{1}{2}ne^{-4rt/n} && (6)\\
&\geq n^{1-\varepsilon} \;, && (7)
\end{aligned}$$

where (6) follows because $(1-p)^{1/p} \geq e^{-2}$ when n is large enough (and therefore p is small enough), and (7) follows from the given condition $rt \leq \frac{1}{8}\varepsilon n \ln n$.

Having established that $Y(p)$ has a high expected value, we now show that it does not drop below that value too often. Let $V(\mathcal{H}) = \{1, 2, \ldots, n\}$. For $1 \leq i \leq n$, let Z_i be the indicator random variable for the event that vertex i is deleted. Then $\Pr[Z_i = 1] = p$. Define the functions $\phi_i : \{0,1\}^i \to \mathbb{R}$ for $0 \leq i \leq n$ and $\psi_i : \{0,1\}^{i-1} \to \mathbb{R}$ for $1 \leq i \leq n$ as follows:

$$\begin{aligned}
\phi_i(z_1, \ldots, z_i) &= \mathrm{E}[Y(p) \mid Z_1 = z_1, \ldots, Z_i = z_i] \;, \\
\psi_i(z_1, \ldots, z_{i-1}) &= \phi_i(z_1, \ldots, z_{i-1}, 0) - \phi_i(z_1, \ldots, z_{i-1}, 1) \;,
\end{aligned}$$

where each $z_j \in \{0,1\}$. It is clear that $\phi_{i-1}(z_1, \ldots, z_{i-1}) = (1-p)\phi_i(z_1, \ldots, z_{i-1}, 0) + p\phi_i(z_1, \ldots, z_{i-1}, 1)$. Therefore,

$$\begin{aligned}
\phi_i(Z_1, \ldots, Z_i) &= \begin{cases} \phi_{i-1}(Z_1, \ldots, Z_{i-1}) + p\psi_i(Z_1, \ldots, Z_{i-1}) \;, & \text{if } Z_i = 0 \\ \phi_{i-1}(Z_1, \ldots, Z_{i-1}) - (1-p)\psi_i(Z_1, \ldots, Z_{i-1}) \;, & \text{if } Z_i = 1 \end{cases} \\
&= \phi_{i-1}(Z_1, \ldots, Z_{i-1}) + (p - Z_i)\psi_i(Z_1, \ldots, Z_{i-1}) \;.
\end{aligned}$$

Using this equation repeatedly, and noting that $\phi_n(Z_1, \ldots, Z_n) = Y(p)$ and that $\phi_0 = \mathrm{E}[Y(p)]$, we get

$$Y(p) \;=\; \mathrm{E}[Y(p)] + \sum_{i=1}^{n} (p - Z_i)\psi_i(Z_1, \ldots, Z_{i-1}) \;.$$

We would like to bound the ranges of the functions ψ_i. To this end, we note that for any fixed $z_1, \ldots, z_{i-1}$, the quantity $\psi_i(z_1, \ldots, z_{i-1})$ is a convex combination of the quantities

$$\phi_n(\underbrace{z_1, \ldots, z_{i-1}}_{\text{fixed}}, 0, \underbrace{z_{i+1}, \ldots, z_n}_{\text{variable}}) - \phi_n(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n) \tag{8}$$

where $(z_{i+1}, \ldots, z_n)$ varies over all tuples in $\{0,1\}^{n-i}$. The quantity (8) is the difference between the number of edges of two subhypergraphs of $\mathcal{H}$ which disagree only at vertex i; clearly, it lies between 0 and d_i, where d_i is the degree of vertex i. Therefore, any convex combination of the quantities (8) also lies between 0 and d_i. Thus, the range of the function ψ_i is contained in $[0, d_i]$.

We would now like to apply Lemma 4.6. Note that $\frac{1}{n}\sum_{i=1}^{n} d_i = \bar{s}$, $\max\{d_1, \ldots, d_n\} = s$, and $pn = 2t$. Thus, Lemma 4.6 gives us

$$\Pr[Y(p) - \mathrm{E}[Y(p)] < -a] \quad \leq \quad \exp\left(\frac{2t\bar{s}}{s}\left(\frac{a}{2t\bar{s}} - \left(1 + \frac{a}{2t\bar{s}}\right)\ln\left(1 + \frac{a}{2t\bar{s}}\right)\right)\right) \quad , \tag{9}$$

We now consider two cases. First, suppose $2t\bar{s} \leq 36s\ln n$. In this case, set $a = 36(e^2 - 1)s\ln n$, which ensures that $\ln(1 + a/(2t\bar{s})) \geq 2$. Therefore, (9) simplifies to

$$\begin{aligned} \Pr[Y(p) - \mathrm{E}[Y(p)] < -36(e^2 - 1)s\ln n] \quad &\leq \quad \exp\left(\frac{2t\bar{s}}{s}\left(-\frac{a}{2t\bar{s}}\right)\right) \\ &= \quad e^{-a/s} \\ &= \quad n^{-36(e^2-1)} \\ &\leq \quad n^{-9/4} \quad . \end{aligned}$$

Using the condition $s \leq n^{1-2\varepsilon}$ and (7), we obtain $\Pr[Y(p) < n^{1-2\varepsilon}] \leq n^{-9/4}$.

For the other case, suppose $2t\bar{s} > 36s\ln n$. Note that $\ln(1 + u) \geq u - u^2/2$, for any $u \geq 0$. We use this inequality, together with the substitution $u = a/(2t\bar{s})$, to simplify (9). After some routine algebra we get

$$\Pr[Y(p) - \mathrm{E}[Y(p)] < -a] \quad \leq \quad \exp\left(-\frac{a^2}{4ts\bar{s}}\left(1 - \frac{a}{2t\bar{s}}\right)\right) \quad .$$

Now we set $a = 3\sqrt{2ts\bar{s}\ln n}$. This makes $a/(2t\bar{s}) < 3\sqrt{1/36} = 1/2$. So,

$$\begin{aligned} \Pr[Y(p) - \mathrm{E}[Y(p)] < -3\sqrt{2ts\bar{s}\ln n}] \quad &\leq \quad \exp\left(-\frac{a^2}{8ts\bar{s}}\right) \\ &\leq \quad \exp\left(-\frac{9\ln n}{4}\right) \\ &= \quad n^{-9/4} \quad . \end{aligned}$$

Using the condition $ts\bar{s} \leq n^{2-3\varepsilon}$ and (7), we again obtain $\Pr[Y(p) < n^{1-2\varepsilon}] \leq n^{-9/4}$.

Thus, in both cases, we have $\Pr[Y(p) < n^{1-2\varepsilon}] \leq n^{-9/4}$. Using (5), we get $\Pr[X(t) < n^{1-2\varepsilon}] \leq 2n^{-9/4} \leq n^{-2}$, which is the desired bound. ∎

5 Proof of the Packing Lemma

We now return to proving Lemma 3.10, our improved packing lemma. Recall that from the hypotheses we already have the following degree conditions on the bipartite graphs G and H we wish to pack:

$$\delta(G) \leq q \quad , \quad \delta(H) \leq q \quad , \quad \Delta_R(G) \leq q^2 \quad , \quad \Delta_L(H) \leq q^2 \quad , \tag{10}$$

where we have set $q = (\alpha n \ln n)^{1/3}$, where α is a small constant to be fixed later. Let us write

$$q = \left(\frac{\varepsilon}{32} n\ln n\right)^{1/3} \quad , \quad \text{where } \varepsilon := 32\alpha \, . \tag{11}$$

We shall assume throughout this section that n is even and large enough.

Definition 5.1 *For a subset W of the vertex set of a graph and integer $k \leq |W|$, let $\mathcal{N}(W)$ denote the neighborhood of W. Let* $\mathrm{top}(W, k)$ *and* $\mathrm{bot}(W, k)$ *denote the subsets of W consisting of, respectively, the k highest and k lowest degree vertices in W. For a vertex x, let $\mathcal{N}(x)$ be defined as $\mathcal{N}(\{x\})$.*

Following Hajnal [Haj91], our first step will be to modify G and H suitably so that even stronger degree conditions hold. Let

$$k \quad = \quad \min\left\{\frac{n}{2}, \frac{n}{4\delta(H)}\right\} \quad . \tag{12}$$

From the hypotheses of the packing lemma, we know that $V_R(G)$ has at least $n/2$ isolated vertices; let V_1 be a set of size $n/2$ of these. Let $V_0 = \mathrm{top}(V_L(G), k)$, $V_2 = \mathrm{bot}(V_L(H), k)$ and $V_3 = \mathcal{N}(V_2) \cup \mathrm{top}(V_R(H), \frac{n}{2} - |\mathcal{N}(V_2)|)$. Let us define graphs G' and H' as follows:

$$G' \;=\; G \setminus (V_0 \cup V_1) \;\; ; \qquad H' = H \setminus (V_2 \cup V_3) \;\; .$$

It follows from the construction above that if G' and H' can be packed then so can G and H. This is because having packed G' and H' we may *arbitrarily* identify the vertices in V_0 with those in V_2 and the vertices in V_1 with those in V_3. Now, to show that G' and H' can be packed, we shall need the degree conditions guaranteed by the following lemma.

Lemma 5.2 *The graphs G' and H' are $(n-k, n/2)$-bipartite graphs with the following properties:*

$$\delta_L(G') \le 2q \;\; , \qquad \delta_R(G') \le 2q \;\; , \qquad \delta_L(H') \le 2q \;\; , \qquad \delta_R(H') \le 2q \;\; ,$$

$$\Delta_L(G') \le 4q^2 \;\; , \qquad \Delta_R(G') \le q^2 \;\; ,$$

$$\Delta_L(H') \le q^2 \;\; , \qquad \Delta_R(H') \le 4q \;\; .$$

Proof: The first four inequalities are obvious from (10) and (12), as are the bounds on $\Delta_R(G')$ and $\Delta_L(H')$. By construction, $|\mathcal{N}(V_2)| \le \sum_{v \in V_2} \deg_H(v) \le \delta(H) \cdot n/(4\delta(H)) = n/4$; therefore V_3 contains at least $n/4$ of the highest degree vertices in $V_R(H)$. Since these vertices are removed to obtain H' we have $\Delta_R(H') \le 4\delta(H) \le 4q$. Similarly, we have $\Delta_L(G') \le 4\delta(H)\delta(G) \le 4q^2$. ∎

We prove that G' and H' can be packed using the probabilistic method: let $\sigma_L : V_L(G') \to V_L(H')$ be a uniform random bijection; we shall show that with positive probability there exists a bijection $\sigma_R : V_R(G') \to V_R(H')$ such that (σ_L, σ_R) is a packing. Let $\Gamma = \Gamma(\sigma_L)$ be a bipartite graph on vertex set $(V_R(G), V_R(H))$ defined as follows: for $x \in V_R(G')$, $y \in V_R(H')$ we have $(x, y) \in E(\Gamma)$ iff $\sigma_L(\mathcal{N}(x)) \cap \mathcal{N}(y) = \emptyset$. It is clear that the required bijection σ_R exists iff the graph Γ has a perfect matching. Our task now is to show that the (random) bipartite graph Γ has a perfect matching with positive probability. The most straightforward way of doing this is to use the following fact, attributed to König, and easily proven using Hall's marriage theorem:

Fact 5.3 *Let K be an (m, m)-bipartite graph such that for every $(x, y) \in V_L(K) \times V_R(K)$, we have $\deg_K(x) + \deg_K(y) \ge m$. Then K has a perfect matching.* ∎

Accordingly, we would like to establish lower bounds on the degrees of vertices in Γ. Hajnal [Haj91] used a similar approach but did not exploit the asymmetry between G' and H'. We, however, shall exploit this asymmetry in a crucial way.

Lemma 5.4 *Let $x \in V_R(G')$ and $y \in V_R(H')$ be arbitrary vertices and let σ_L and Γ be as defined above. Let $m = \frac{1}{2}n$. Then, for α (and hence, ε) small enough,* $\Pr[\deg_\Gamma(x) < m^{1-2\varepsilon}] \le m^{-2}$, *and* $\Pr[\deg_\Gamma(y) < m - m^{1-2\varepsilon}] \le m^{-2}$.

Proof: Any bipartite graph K can be recast as a hypergraph $\mathcal{H}_K$ in the following natural way: we let the vertices in $V_L(K)$ be the vertices of $\mathcal{H}_K$ and the neighborhoods of vertices in $V_R(K)$, possibly repeated, form the multiset of hyperedges. Formally, $V(\mathcal{H}_K) = V_L(K)$, and $E(\mathcal{H}_K) = \{\{\mathcal{N}(x) : x \in V_R(K)\}\}$. Construct the hypergraphs $\mathcal{H}_{G'}$ and $\mathcal{H}_{H'}$, from G' and H' respectively, in this manner. Then each of these hypergraphs has $n - k$ vertices and $m = \frac{1}{2}n$ hyperedges, and is squarish, because $k \le \frac{1}{2}n$. It is clear from Lemma 5.2 that $\mathcal{H}_{G'}$ is $(q^2, 4q^2, 2q)$-favorable and that $\mathcal{H}_{H'}$ is $(4q, q^2, 2q)$-favorable.

Let $S(y) = \sigma_L^{-1}(\mathcal{N}(y)) \subseteq V_L(G')$. The neighbors of y in Γ are precisely those vertices in $V_R(G')$ whose neighborhoods in G' do not intersect $S(y)$. Thus, $\deg_\Gamma(y)$ is the number of hyperedges that survive upon deleting the vertices in $S(y)$ from the hypergraph $\mathcal{H}_{G'}$, i.e., $|E(\mathcal{H}_{G'} \setminus S(y))|$. But $S(y)$ is a uniform random subset of $V_L(G')$ of size $|\mathcal{N}(y)| \le \Delta_R(H') \le 4q$. Choose α (and hence, ε) small enough so that the conditions of Lemma 4.4 hold: $4q^2 \le n^{1-3\varepsilon}$ and $4q \cdot 2q \le n^{1-3\varepsilon}$; our choice of q in (11) ensures that this can be done. That lemma then implies $\Pr[\deg_\Gamma(y) < m - m^{1-2\varepsilon}] \le m^{-2}$.

Let $T(x) = \sigma_L(\mathcal{N}(x)) \subseteq V_L(H')$. Reasoning as above, $\deg_\Gamma(x) = |E(\mathcal{H}_{H'} \setminus T(x))|$. But $T(x)$ is a uniform random subset of $V_L(H')$ of size $|\mathcal{N}(x)| \le \Delta_R(G') \le q^2$. Choose α small enough so that the conditions of Lemma 4.5 hold: $q^2 \le n^{1-2\varepsilon}$, $4q \cdot q^2 \le \frac{1}{8}\varepsilon n \ln n$, and $q^2 \cdot q^2 \cdot 2q \le n^{2-3\varepsilon}$; our choice of q in (11) ensures that this can be done. *We remark that the second of these three conditions is the bottleneck that limits the strength of the packing lemma, and thus, of the main theorem.* Lemma 4.5 then implies that $\Pr[\deg_\Gamma(x) < m^{1-2\varepsilon}] \le m^{-2}$. ∎

Corollary 5.5 *Let σ_L and Γ be as defined above and α be as small as required in Lemma 5.4. Then, with positive probability, Γ has a perfect matching. Therefore, G' and H' can be packed.*

Proof: Let $m = n/2$. Then Γ is an (m, m)-bipartite graph. Lemma 5.4 implies that any one of the $2m$ "bad" events $\deg_\Gamma(x) < m^{1-2\varepsilon}$ for $x \in V_R(G') = V_L(\Gamma)$ and $\deg_\Gamma(y) < m - m^{1-2\varepsilon}$ for $y \in V_R(H') = V_R(\Gamma)$ happens with probability at most m^{-2}. Applying the union bound, the probability that no bad event occurs is at least $1-(2m)/m^2 > 0$. Thus, with positive probability, Γ is such that for every choice of $x \in V_L(\Gamma)$ and $y \in V_R(\Gamma)$,

$$\deg_\Gamma(x) + \deg_\Gamma(y) \quad \geq \quad m^{1-2\varepsilon} + (m - m^{1-2\varepsilon}) \quad = \quad m \quad .$$

Therefore, by Fact 5.3, Γ has a perfect matching. ∎

This concludes the proof of Lemma 3.10, the packing lemma.

6 Proof of the Main Theorem

We wrap up by putting together the complete chain of reasoning that establishes Theorem 1.1, the main theorem.

Proof of Theorem 1.1: Let $P \in \mathscr{P}_{n,n}$ be a bipartite graph property. Set $q = (\alpha n \ln n)^{1/3}$, where α is chosen small enough for the packing lemma to hold. We claim that $\mathcal{C}^R(P) = \Omega(nq)$.

Suppose not. Assume n to be even, without loss of generality. Let G be an R-first sparse minterm of P (essentially guaranteed to exist by Lemma 3.6) and H be an L-first minterm of P^*. As explained in Section 3, Fact 3.7 then implies $\delta(G) \leq q$, $\delta(H) \leq q$, $\Delta_R(G) \leq q^2$, and $\Delta_L(H) \leq q^2$. Therefore, by Lemma 3.10 (our improved packing lemma), G and H can be packed. This contradicts Fact 3.9, which says that G and H cannot be packed.

Thus, we have proved by contradiction that $\mathcal{C}^R(P) = \Omega(nq)$, for all $P \in \mathscr{P}_{n,n}$. Since $nq = O(n^{3/2})$, Fact 3.2 implies that $\mathcal{C}^R(Q) = \Omega(nq) = \Omega(n^{4/3} \log^{1/3} n)$ for any $Q \in \mathscr{P}_n$. ∎

As a concluding remark, we note that there is plenty of slackness in the parameters of Lemmas 4.4 and 4.5, the hypergraph vertex deletion lemmas. It is only Hajnal's framework, which we use here, that constrains the parameters we are forced to use those lemmas with. This raises the open question of whether the vertex deletion lemmas have other interesting applications in graph packing. It also suggests that further improvement of our $\Omega(n^{4/3} \log^{1/3} n)$ lower bound will require breaking out of Hajnal's framework.

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References

[AS00] Noga Alon and Joel H. Spencer. *The Probabilistic Method*. Wiley-Interscience, New York, 2000.

[Bol78] Béla Bollobás. *Extremal Graph Theory*. Academic Press, New York, NY, 1978.

[CKS02] Amit Chakrabarti, Subhash Khot, and Yaoyun Shi. Evasiveness of subgraph containment and related properties. *SIAM J. Comput.*, 31(3):866–875, 2002.

[DK00] Ding-Zhu Du and Ker-I Ko. *Theory of Computational Complexity*. John Wiley and Sons, New York, NY, 2000.

[FKW02] Ehud Friedgut, Jeff Kahn, and Avi Wigderson. Computing graph properties by randomized subcube partitions. In *Proc. 6th International Workshop on Randomization and Approximation Techniques in Computer Science*, pages 105–113, 2002.

[Grö92] Hans Dietmar Gröger. On the randomized complexity of monotone graph properties. *Acta Cybernetica*, 10:119–127, 1992.

[Haj91] Péter Hajnal. An $\Omega(n^{\frac{4}{3}})$ lower bound on the randomized complexity of graph properties. *Combinatorica*, 11:131–143, 1991.

[Hoe63] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Stat. Assoc.*, 58(301):13–30, 1963.

[HS92] Péter Hajnal and Mario Szegedy. On packing bipartite graphs. *Combinatorica*, 11:295–301, 1992.

[Kin88] Valerie King. Lower bounds on the complexity of graph properties. In *Proc. 20th Annual ACM Symposium on the Theory of Computing*, pages 468–476, 1988.

[KSS84] J. Kahn, M. Saks, and D. Sturtevant. A topological approach to evasiveness. *Combinatorica*, 4:297–306, 1984. Preliminary version in *Proc. 24th Annu. IEEE Symp. Found. Comp. Sci.*, pages 31–39, 1983.

[RV76] Ronald L. Rivest and Jean Vuillemin. On recognizing graph properties from adjacency matrices. *Theor. Comput. Sci.*, 3(3):371–384, 1976.

[Yao87] Andrew C. Yao. Lower bounds to randomized algorithms for graph properties. In *Proc. 28th Annual IEEE Symposium on Foundations of Computer Science*, pages 393–400, 1987.

[Yao88] Andrew C. Yao. Monotone bipartite graph properties are evasive. *SIAM J. Comput.*, 17(3):517–520, 1988.