Homework 1
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General Instructions

- Please typeset your solutions using either LaTeX or TeX. Submit a printout of your solution *before* class.
- You may find this homework hard. Don't worry if you cannot solve every problem perfectly.
- You may discuss the problems with your classmates, but you must write up your solutions on your own, in your own words. If you borrowed a key idea from someone else, please acknowledge them like a true scholar.
- You may refer to both textbooks and to all of the papers linked from the course web site. You may refer to math textbooks or the web site `mathworld.wolfram.com` to look up any math you may have forgotten. But referring to anything else (other web sites, other published papers, etc.) is a *violation of the honor code*.
- Make sure your solutions are complete, clean, concise and rigorous. Please do not submit a solution that you know to be erroneous or incomplete in the hope that you may get some partial credit. You will not.
- Please think carefully about how you are going to organise your solutions *before* you begin writing.

The Problems

Note: [MR] refers to the Motwani-Raghavan book and [CLRS] to the Cormen-Lieserson-Rivest-Stein book.

1. Solve Problem 8.21 from [MR]. [15 points]

2. Solve Problems 11-4(a) and 11-4(b) from [CLRS]. ***Important:*** [CLRS] uses the term 2-universal differently from [MR], so keep this in mind as you solve this problem. In short:

$$\begin{aligned}\text{[CLRS] universal} &\quad\longmapsto\quad \text{[MR] } 2\text{-universal}\,,\\ \text{[CLRS] } 2\text{-universal} &\quad\longmapsto\quad \text{[MR] strongly } 2\text{-universal}\,.\end{aligned}$$

It turns out that different printings of [CLRS] have different version of the problem. Please solve *this* problem:

Let $\mathcal{H} = \{h\}$ be a class of hash functions in which each h maps the universe U of keys to $\{0, 1, \ldots, m-1\}$. We say that $\mathcal{H}$ is ***k-universal*** if, for every fixed sequence of k distinct keys $\langle x^{(1)}, x^{(2)}, \ldots, x^{(k)}\rangle$ and for any h chosen at random from $\mathcal{H}$, the sequence $\langle h(x^{(1)}), h(x^{(2)}), \ldots, h(x^{(k)})\rangle$ is equally likely to be any of the m^k sequences of length k with elements drawn from $\{0, 1, \ldots, m-1\}$.

a. Show that if $\mathcal{H}$ is 2-universal, then it is universal.

b. Let U be the set of n-tuples of values drawn from $\mathbf{Z}_p$, and let $B = \mathbf{Z}_p$, where p is prime. For any n-tuple $a = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ of values from $\mathbf{Z}_p$ and for any $b \in \mathbf{Z}_p$, define the hash function $h_{a,b} : U \to B$ on an input n-tuple $x = \langle x_0, x_1, \ldots, x_{n-1}\rangle$ from U as

$$h_{a,b}(x) \;=\; \left(\sum_{j=0}^{n-1} a_j x_j + b\right) \bmod p$$

and let $\mathcal{H} = \{h_{a,b}\}$. Argue that $\mathcal{H}$ is 2-universal. [10+10 points]

3. Suppose T is an n-node binary search tree with average access time $\Theta(\log n)$ under a uniform distribution (for the accesses) over the items in the tree. Look at page 673 of the Sleator-Tarjan paper for the definition of average access time.

   3.1. Prove that the worst case cost of splaying at a node in T is $O(\sqrt{n \log n})$. [10 points]

   3.2. Prove that the above bound is tight via a concrete example. [5 points]

4. Describe a sequence of accesses to an n-node splay tree T, where n is odd, that results in T consisting of a single chain of nodes that alternates between left children and right children. In other words, T, after the accesses, should look like a saw blade. Argue that your sequence gives the desired result.

   You might consider gaining some intuition by playing with an interactive splay tree applet (this is an exception to the honor code rule which forbids you from looking for solutions on general web sites). Google is your friend. [20 points]

5. Solve Problems 10.9 through 10.11 from [MR]. Note that these problems speak of *multigraphs* rather than weighted graphs. Adjust your wording accordingly. [8+8+4 points]

6. Solve Exercise 10.9 (page 293) in [MR]. [10 points]