12. For a string $x \in \{0,1\}^n$, let $N_1(x)$ denote the number of 1s in $x$. The majority function $\text{MAJ}_n : \{0,1\}^n \to \{0,1\}$ is defined as follows:

$$\text{MAJ}_n(x) = \begin{cases} 
1, & \text{if } N_1(x) \geq n/2, \\
0, & \text{otherwise}.
\end{cases}$$

Show that $\text{MAJ}_n$ can be computed using $O(n)$-sized circuits. [This is essentially Sipser's Problem 9.26 — if you use the approach suggested in the book, you need to first solve (in sufficient detail) any subproblems that come up, such as Sipser's Problem 9.24.] [2 points]

13. Prove that Shannon's lower bound is tight up to constant factors. That is, improve the upper bound we showed in class by proving that every function $f : \{0,1\}^n \to \{0,1\}$ has an $n$-input circuit of size $O(2^n/n)$. [2 points]

The second problem is hard. A hint is to consider the function $f$ as being $f(y,z)$, where $y = \{x_1,\ldots,x_k\}$ and $z = \{x_{k+1},\ldots,x_n\}$. Now, the truth table of $f$ can be viewed as a $2^k \times 2^{n-k}$ matrix, with the rows indexed by all possible assignments to $y$. Each column of this matrix gives us a certain pattern in $\{0,1\}^{2^k}$. What if there aren't too many different patterns? Can we use that fact to cut down on the circuit size?