Solutions to Quiz 1
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If you have any questions about the grading of a particular problem, please consult the appropriate grader as shown below:

- Problems 1, 2, 3 were graded by Chien-Chung Huang.
- Problems 4, 5, 6 were graded by David Blinn.
- Problems 7, 8, 9 were graded by Amit Chakrabarti.

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1. State the product principle, in any form you like. [5 points]

   Solution: The size of the union of m pairwise disjoint sets, each of size n, equals mn.
   OR
   If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

   Note: You must point out that the sets are *pairwise disjoint* or *mutually disjoint*.

2. At John Smith's dinner party, every invitee was either a Dartmouth student or had grown up in New Hampshire. There were 32 inivitees who were Dartmouth students and 25 who had grown up in New Hampshire. There were 8 invitees who were both Dartmouth students *and* had grown up in New Hampshire. How many invitees attended John Smith's party? [10 points]

   Solution: We have two sets: one of invitees who were Dartmouth students and one of invitees who had grown up in New Hampshire. Adding up their sizes counts each invitee who was both a Dartmouth student and had grown up in New Hampshire *twice*. Therefore, to compensate, we must subtract the number of these special invitees *once*. So, the answer is $32 + 25 - 8 = 49$.

3. Suppose A and B are finite sets. Write an equation connecting the the sizes of the four sets A, B, $A \cup B$ and $A \cap B$ that captures the abstract essence of the previous problem. [5 points]

   Solution: $|A \cup B| = |A| + |B| - |A \cap B|$.

4. Give a complete and *mathematically precise* definition of a "symmetric relation." You may assume that the reader knows the definition of "relation." [10 points]

Solution: A symmetric relation on a set A is a relation R such that

$$\forall\, a, b \in A \;\; ((a,b) \in R \Rightarrow (b,a) \in R)\,.$$

Note: You must say that the relation is "on a set A" or "from A to A" or "a subset of $A \times A$." You must also quantify a and b using the universal quantifier "$\forall$" (it's also okay to write "for all" or "for every" in English). These details are important in a mathematically precise definition.

5. Suppose p, q and r are propositions. Draw a truth table for the compound proposition $p \Rightarrow (q \Rightarrow r)$. Your truth table should have 5 columns labeled as follows:

$$p, \quad q, \quad r, \quad q \Rightarrow r, \quad p \Rightarrow (q \Rightarrow r)\,.$$

[10 points]

Solution:

p	q	r	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Note how this truth table reveals that $p \Rightarrow (q \Rightarrow r)$ is the same thing as $\neg p \vee \neg q \vee r$.

6. Evaluate $\binom{10}{7}$. [5 points]

Solution:

$$\binom{10}{7} = \binom{10}{3} = \frac{10^{\underline{3}}}{3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 10 \times 3 \times 4 = 120\,.$$

7. Find integers n and k such that $\binom{17}{9} + 2\binom{17}{10} + \binom{17}{11} = \binom{n}{k}$. [10 points]

Solution: Using Pascal's relation three times, we get

$$\begin{aligned}
\binom{17}{9} + 2\binom{17}{10} + \binom{17}{11} &= \left[\binom{17}{9} + \binom{17}{10}\right] + \left[\binom{17}{10} + \binom{17}{11}\right] \\
&= \binom{18}{10} + \binom{18}{11} \\
&= \binom{19}{11}\,.
\end{aligned}$$

Thus, $n = 19$ and $k = 11$.

If you want to add an extra twist, $k = 8$ also works, because $\binom{19}{11} = \binom{19}{8}$.

8. Every bicycle on the campus of Sunapee College is required to have a 3-digit registration code from the Security Office. A registration code ranges from 000 to 999, inclusive. The students there are superstitious; they consider any registration code containing the digit '8' to be unlucky, because of the resemblance of this digit to the mangled wheels of a crashed bicycle. How many registration codes are unlucky? [10 points]

Solution: Let's count the number of registration codes that are *not* unlucky. These codes avoid '8' altogether, so there are only 9 (not 10) possibilities for each of the three digit positions. By the product principle, there are $9 \times 9 \times 9 = 729$ such codes.

The total number of possible codes is 1000. By the sum principle, 729 plus the number of unlucky codes equals 1000. Therefore there are $1000 - 729 = 271$ unlucky codes.

9. Using mathematical induction, prove that for all integers $n \geq 1$,

$$3^0 + 3^1 + 3^2 + \cdots + 3^{n-1} \;=\; \frac{3^n - 1}{2}\,.$$

[15 points]

Solution: Let S_n denote the proposition "$\sum_{i=0}^{n-1} 3^i \;=\; \frac{1}{2}(3^n - 1)$." We wish to prove that $\forall n \geq 1\,(S_n)$. We shall use induction on n.

BASE CASE, S_1: The proposition S_1 reads "$3^0 = \frac{1}{2}(3^1 - 1)$." This simplifies to "$1 = 1$," which is obviously true.

INDUCTION STEP, $\forall k \geq 1\,(S_k \Rightarrow S_{k+1})$: Let us make the inductive hypothesis that S_k is true. Then

$$\begin{aligned}
3^0 + 3^1 + 3^2 + \cdots + 3^{k-1} + 3^k &= \frac{3^k - 1}{2} + 3^k \qquad \text{[by the inductive hypothesis]} \\
&= \frac{3^k - 1 + 2 \times 3^k}{2} \\
&= \frac{(1+2) \times 3^k - 1}{2} \\
&= \frac{3 \times 3^k - 1}{2} \\
&= \frac{3^{k+1} - 1}{2}\,.
\end{aligned}$$

Therefore S_{k+1} is true. QED.