Solutions to Quiz 2
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If you have any questions about the grading of a particular problem, please consult the appropriate grader as shown below:

- Problems 1 and 2 were graded by David Blinn.
- Problem 3 was graded by Chien-Chung Huang.
- Problems 4 and 5 were graded by Amit Chakrabarti.

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1. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of all non-negative integers and let $\mathbb{R}^+$ be the set of all non-negative real numbers. Suppose we have two functions $f, g : \mathbb{N} \to \mathbb{R}^+$. Write precise mathematical definitions of the notations $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

   Solution: $f(n) = O(g(n))$ means $\exists\, c, n_0 > 0\ (\forall\, n \geq n_0\ (f(n) \leq cg(n)))$.
   $f(n) = \Omega(g(n))$ means $\exists\, c, n_0 > 0\ (\forall\, n \geq n_0\ (f(n) \geq cg(n)))$.

2. Let S be a sample space for a random process and let P be the appropriate probability distribution on S. Answer the following questions with precise mathematical definitions.

   2.1. What does it mean to say that P is a uniform probability distribution?

   Solution: Any of the following would be an appropriate definition:
   $\forall\, x \in S\ (P(x) = 1/|S|)$.

   OR

   $\forall\, x, y \in S\ (P(x) = P(y))$.

   OR

   P assigns equal weight to all elements of S.

   2.2. Let $A, B \subseteq S$ be two events. What does it mean to say that A and B are independent?

   Solution: It means $P(A \mid B) = P(A)$.

3. For each of the recurrences below, find a big-Θ bound on the solution. You may use theorem(s) and results(s) from the textbook without proof. If you ever need to use the fact that $f(n) = \Theta(g(n))$ for some f and g, you should state this, but you need not prove it.

3.1.

$$T(n) \;=\; \begin{cases} 4T(\lceil n/2 \rceil) + 2n^2 - 9n + 6, & \text{if } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

Solution: We have $2n^2 - 9n + 6 = \Theta(n^2)$. We also have $\log_2 4 = 2$. The master theorem tells us that $T(n) = \Theta(n^2 \log n)$.

3.2.

$$T(n) \;=\; \begin{cases} 27T(\lceil n/3 \rceil) + n^3, & \text{if } n > 1 \\ 2, & \text{if } n = 1. \end{cases}$$

Solution: The initial condition $T(1) = 2$ doesn't matter; we have $\log_3 27 = 3$ and so the master theorem tells us that $T(n) = \Theta(n^3 \log n)$.

3.3.

$$T(n) \;=\; \begin{cases} 4T(\lceil n/3 \rceil) + \dfrac{n^2}{\sqrt{n+1}}, & \text{if } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

Solution: We have $\frac{n^2}{\sqrt{n+1}} = \Theta(n^{3/2})$, so we want to compare $\log_3 4$ with $3/2$. Since $3^{3/2} = \sqrt{27} > \sqrt{16} = 4$, we have $3/2 > \log_3 4$. Now the master theorem tells us that $T(n) = \Theta(n^{3/2})$.

3.4.
$$T(n) \;=\; \begin{cases} 4T(\lceil n/2 \rceil) + n \log n, & \text{if } n > 1 \\ 2, & \text{if } n = 1. \end{cases}$$

Solution: Let us define two new sequences $L(n)$ and $U(n)$ as follows: $L(1) = U(1) = 2$, and for $n > 1$,

$$\begin{aligned} L(n) &= 4L(\lceil n/2 \rceil) + n\,, \\ U(n) &= 4U(\lceil n/2 \rceil) + n^{3/2}\,. \end{aligned}$$

Then, for all n, $L(n) \le T(n) \le U(n)$ (this can be proved rigorously by induction). But the master theorem tells us that $L(n) = \Theta(n^2)$ and $U(n) = \Theta(n^2)$. Therefore $T(n) = \Theta(n^2)$ as well.

This problem was almost identical to textbook P4.5-7 from the textbook. The hint at the back of the textbook suggests a different, but more long-winded, approach. (Although, why on earth would one want to work harder than necessary?!)

4. As discussed in class, a virtual 3-sided die that shows the numbers x_1, x_2 and x_3 with equal probability can be constructed out of a usual 6-sided die by writing each of the x_i on two of the faces. Suppose that we construct three specific 3-sided dice as follows:

- The red die shows the numbers in {3,5,7} with equal probability.
- The green die shows the numbers in {2,4,9} with equal probability.
- The blue die shows the numbers in {1,6,8} with equal probability.

We say that "die A beats die B" if, upon rolling both dice, the number on top of A exceeds the number on top of B with probability $> 50\%$.

4.1. Supposed the red die and the green die are both rolled. Let E_{RG} denote the event that the number on top of the red die exceeds the number on top of the green die. Compute $P(E_{RG})$. Does the red die beat the green die?

Solution: A sample space for rolling these two dice is given by

$$S_{RG} \;=\; \{(i,j) : \text{ the red die shows } i \text{ and the green die shows } j\}\,.$$

Since the dice are fair, the probability distribution on this sample space is uniform. We see that $|S_{RG}| = 9$ and that $E_{RG} = \{(3,2),(5,2),(5,4),(7,2),(7,4)\}$, so that $|E_{RG}| = 5$. Therefore, using the formula for the probability of an event under a uniform distribution, we have

$$P(E_{RG}) \;=\; \frac{|E_{RG}|}{|S_{RG}|} \;=\; \frac{5}{9} \;>\; 50\%\,,$$

which means that the red die *does beat* the green die.

4.2. Does the green die beat the blue die? Show your work!

Solution: Proceeding as above, the situation is modelled by a uniform distribution on the sample space

$$S_{GB} \;=\; \{(i,j) : \text{the green die shows } i \text{ and the blue die shows } j\}$$

and the event $E_{GB} = \{(2,1),(4,1),(9,1),(9,6),(9,8)\}$. Therefore

$$P(E_{GB}) \;=\; |E_{GB}|/|S_{GB}| \;=\; 5/9 \;>\; 50\%$$

and we conclude that the green die *does beat* the blue die.

4.3. Does the blue die beat the red die? Again, show your work.

Solution: A similar calculation gives a probability of $5/9$ again and shows that (wonder of wonders) the blue die *does beat* the red die! Yet another way in which probability can be non-intuitive.

5. A standard deck of 52 cards is randomly shuffled and separated into four bridge hands (traditionally called North, East, South and West) of 13 cards each. A little birdie tells you that North and South hold 11 spades between them, and thus, East and West hold 2 spades between them. Given this information, what is the probability that East and West hold *exactly one* spade each?

   Free hint: The answer is *not* 50%.

   Solution: Let S be the set of cards held by East and West. It does not matter exactly which cards are in S, just that $|S| = 26$ and S contains exactly two spades. Now, we divide S into two hands of 13 cards each. The given question is: what is the probability that both East and West end up with one spade each?

   There are $\binom{26}{13}$ ways to divide S into two hands, all equally likely. Of these, let us count the number of divisions in which East gets exactly one spade. We can choose which spade (s)he gets in $\binom{2}{1}$ ways and which 12 non-spades (s)he gets in $\binom{24}{12}$ ways; by the product principle, this gives a total of $\binom{2}{1}\binom{24}{12}$ ways. Therefore,

$$\begin{aligned}\text{Required probability} &= \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \\ &= \frac{2 \cdot \frac{24!}{12! \cdot 12!}}{\frac{26!}{13! \cdot 13!}} \\ &= \frac{2 \cdot 24! \cdot 13! \cdot 13!}{26! \cdot 12! \cdot 12!} \\ &= \frac{2 \cdot 13 \cdot 13}{26 \cdot 25} \\ &= \frac{13}{25}\,.\end{aligned}$$

   Thus, the desired answer is $13/25 = 52\%$.