# Homework 1 (Grader: Blinn)
## Due Jan 18, 2006

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In this and all future homeworks in this course, you must demonstrate how you arrived at your final answers — i.e., you must show your steps — unless the problem statement makes an exception. You must also justify any steps that are not trivial. Please think carefully about how you are going to organise your answers *before* you begin writing.

You may want to refer to the slides posted on the course website as you work on these problems. You may also want to refer occasionally to Chapter 3 of the textbook, for the problems on logic.

Throughout this homework, $\mathbb{Z}$ refers to the set of all integers.

1. **(Roster and Set-Builder Notations)** Here are some sets described in set-builder notation. Describe each of them in roster notation. You do not need to show any steps.

   1.1. $\{x^2 : x \in \mathbb{Z} \text{ and } x^2 < 20\}$ [2 points]

   1.2. $\{k \in \mathbb{Z} : 10 \le k \le 99 \text{ and the sum of the digits of } k \text{ is } 9\}$ [2 points]

   1.3. $\{x \in \mathbb{Z} : 0 \le x \le 10 \text{ and } \frac{x}{2} \notin \mathbb{Z}\}$ [2 points]

   1.4. $\{S : S \subseteq \{a, b\}\}$ [2 points]

   1.5. $\{S : S \subseteq \{a, b, c, d\} \text{ and } |S| \text{ is even}\}$ [4 points]

   1.6. $\{S : \{1, 2\} \subseteq S \subseteq \{1, 2, 3, 4\}\}$ [4 points]

   1.7. $\{S \subseteq \{1, 2, 3, 4\} : S \text{ is disjoint from } \{2, 3\}\}$ [4 points]

2. **(Operations on Sets)** Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8, 10\}$ and $C = \{0, 1, 5, 6, 9\}$. In the following subproblems, you must show your steps for those cases where the statement asks you to "verify" an equation. For the rest, you do not need to show any steps.

   2.1. What is $A \cup B$? What is $(A \cup B) \cup C$? [2 points]

   2.2. What is $B \cup C$? What is $A \cup (B \cup C)$? [2 points]

   2.3. What is $A \cap B \cap C$? [2 points]

   2.4. Verify by direct computation that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. [4 points]

   2.5. What is $A - B$? What is $B - C$? [2 points]

   2.6. What is $(A - B) - C$? What is $A - (B - C)$? [4 points]

2.7. Verify by direct computation that $(A - B) - C = A - (B \cup C)$. [5 points]

2.8. Verify by direct computation that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$. [5 points]

2.9. What is $(A \cap B) \times (B - C)$? [4 points]

3. **(Relations)** Let $S = \{1, 3, 5, 7, 9\}$ and $T = \{0, 2, 4, 6, 8\}$. Let's say that an element $x \in S$ "completes" an element $y \in T$ if $x + y$ is divisible by $3$. Describe the relation "completes" from $S$ to $T$ as a subset of $S \times T$ (i.e., write out all the pairs in this relation). Then describe the same relation pictorially, using arrows, as done in class. [5 points]

4. **(Special types of relations)** For each of the following relations, state whether or not it is (a) reflexive, (b) symmetric, (c) antisymmetric and (d) transitive. Whenever your answer is "no", explain why. This means that if, for instance, you say that a relation $R$ is not symmetric, you must exhibit a pair $(a, b)$ such that $(a, b) \in R$ but $(b, a) \notin R$.

4.1. The relation "divides", on the set of all positive integers. (Note: we say that $m$ divides $n$ if $n/m$ is an integer.) [5 points]

4.2. The relation "is disjoint from", on $\mathcal{P}(\mathbb{Z})$. (Note: $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$.) [5 points]

4.3. $\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m > 0, n > 0$ and the sum of the digits of $m$ equals the sum of the digits of $n\}$. [5 points]

5. **(Functions)** Suppose $f : A \to B$ is a function. Define the relation $f^{-1}$ from $B$ to $A$ as follows:

$$f^{-1} \;=\; \{(y, x) \in B \times A : \; f(x) = y\}\,.$$

5.1. Suppose $f^{-1}$ is also a function. What can you then conclude about $f$ with regard to surjectivity and injectivity? [6 points]

5.2. Does the implication work the other way, i.e., given that $f$ has the properties that you came up with, above, does it follow that $f^{-1}$ is a function? Briefly explain why or why not. [4 points]

By the way, the notation $f^{-1}$ is pronounced "$f$ inverse".

6. **(Logic)** Let $p$ and $q$ denote arbitrary propositions.

6.1. Write out a truth table for the compound proposition $\neg(p \Rightarrow q)$. Your table should have $4$ columns: one for $p$, one for $q$, one for $p \Rightarrow q$ and finally one for your target $\neg(p \Rightarrow q)$. [5 points]

6.2. Write out a truth table for $(p \Rightarrow \neg q)$. Is $\neg(p \Rightarrow q)$ logically equivalent to $(p \Rightarrow \neg q)$? [5 points]

6.3. Using a truth table, verify that $\neg(p \Rightarrow q)$ is logically equivalent to $(p \wedge \neg q)$. Does this make intuitive sense to you? Give an explanation in plain English why the opposite of the statement "$p$ implies $q$" is the statement "$p$ and not $q$". [10 points]