CS19: Solutions to Homework 2

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You must demonstrate how you arrived at your final answers — i.e., you must show your steps — unless the problem statement makes an exception. You must also justify any steps that are not trivial. Simply writing down a final answer *will not earn any credit*. Please think carefully about how you are going to organise your answers *before* you begin writing.

The notation P$i.j$-k refers to Problem k from the list of problems after Section $i.j$ in your textbook. Thus, P1.2-4 refers to Problem 4 on page 17.

1. Solve P1.2-4.

   Solution:

   If we list S as $x_1, x_2, \ldots, x_s$, then there is a bijection between functions from S to T and lists $f(x_1), f(x_2), \ldots f(x_s)$. For each i there are t choices for $f(x_i)$. So by the product principle, there are t^s functions from S to T.

2. Solve P1.2-6.

   Solution:

   This problem translates into deciding which children are to get a piece of fruit. We are asking for the number of k-element subsets of n children, which is $\binom{n}{k}$, and is zero if $k > n$.

3. Solve P1.2-7; note that a "five-digit number" must lie between 10000 and 99999 inclusive.

   Solution:

   First, note that "a five digit (base 10) number" means a string of five digits, where the first digit is not "0" and each digit is in the set $0, 1, \ldots, 9$.

   By the product rule, the number of these is $9 * 10^4$, i.e. ninety thousand.

   If no two consecutive digits can be equal, then there are 9 choices for the first digit, 9 for the second (any digit other than the first), 9 for the third (any digit other than the second), and so on. By the product rule, the total number is $9^5 = 59049$.

   By sum rule, the total number of five digit numbers, equals the number that have no two consecutive digits equal, plus the number that have at least one pair of consecutive digits equal. Thus, letting x denote the number of the latter, we have $9 * 10^4 = 9^5 + x$, so $x = 9 * 10^4 - 9^5 = 30951$.

4. Solve P1.2-8.

   Solution:

   In both cases there 2 ways to decide whether the leftmost spot is one for a student or one for an administrator. This decision determines which four places are for students and which are for administrators. Then there are 4! ways to assign the students to their places and 4! ways to assign the administrators to their places. Thus in both cases the product principle leads us to conclude that there are $2 \cdot 4! \cdot 4!$ lists.

5. Solve P1.2-13.

   Solution:

   We assume that both scoops can be of any flavor. When both scoops are of the same flavor, we have 10 possibilities, since there are only 10 flavors. When both scoops have different flavors, we have 45 possibilities according to mother's rule (that the order of the scoops does not matter - so we are finding the number of subsets of 2 flavors out of ten flavors). So, for ice cream, we have $10+\binom{10}{2} = 10+45 = 55$ possibilities.

   For topping, we have 3 possibilities. For whipped cream, nuts and cherry, we make a binary choice - either to have the topping or not, so we have two possibilities for each of them, so by product rule, we have $2 \times 2 \times 2 = 8$ possibilities. So we have $55 \times 3 \times 8 = 1320$ possible sundaes.

6. Solve P1.2-15.

   Suppose we list the people in the club in some way, and keep that list for the remainder of the problem. Take the first person from the list and pair that person with any of the $2n-1$ remaining people. Now take the next unpaired person from the list and pair that person with any of the remaining $2n-3$ unpaired people. Continuing in this way, once k pairs have been selected, take the next unpaired person from the list and pair that person with any of the remaining $2n-2k-1$ unpaired people. Every pairing can arise in this way, and no pairing can arise twice in this process. Thus the number of outcomes is $\prod_{i=0}^{n-1}(2n-2i-1)$.

   For another solution, choose people in pairs. There are $\binom{2n}{2}$ ways to choose one pair, $\binom{2n-2}{2}$ ways to choose a second pair, and once k pairs have been chosen, there are $\binom{2n-2k}{2}$ ways to choose the next pair. The number of lists of pairs we get in this way is $\prod_{i=0}^{n-1}\binom{2n-2i}{2} = \frac{(2n)!}{2^i}$. However each way of pairing people gets listed $n!$ times since we see all possible length n lists of pairs. Therefore the number of actual pairings is $\frac{(2n)!}{2^n n!} = \frac{2n!}{2n \cdot 2n-2 \cdot 2n-4 \cdot \ldots \cdot 2} = \prod_{i=0}^{n=1}(2n-2i-1)$.

   In the case where we additionally specify who serves first, we have n pairs with two options each for who serves first. By the product rule, this gives 2^n additional possibilities to specify the pairs. Thus, there are a total of $2^n \prod_{i=0}^{n-1}(2n-2i-1)$ ways in which to specify the pairs in this case.

7. Let $f : A \to B$ be a function and suppose $|A| = |B| = n$. Prove that

   $$f \text{ is injective } \iff f \text{ is surjective.}$$

   Note: The above problem asks you to "explain" a fact, i.e., write a proof of a certain statement. In this case, the statement is an "if and only if" statement, so your proof should be neatly divided into two clearly marked parts, organized as follows. In the first part, prove the "$\Longrightarrow$" direction: start by assuming that f is injective. Then make a sequence of logical deductions that end in the conclusion that f is surjective. *Avoid writing a lengthy paragraph or overlong sentences!* Stick to short, crisp sentences. In the second part, prove the "$\Longleftarrow$" direction: start by assuming that f is surjective and eventually conclude that f is injective.

   You can title the two parts as you like. One convention simply titles them "$\Longrightarrow$" and "$\Longleftarrow$".

   Solution:

   $$f \text{ is injective } \Longrightarrow f \text{ is surjective.}$$

   Direct proof: That f is injective tells us that $\forall x, y \in A$ if $x \neq y, f(x) \neq f(y)$. Since f is a function, $\forall x \in A, f(x) \in B$. Thus, the n elements of A, when given as inputs to f, must produce n different outputs in B. Given that $|B| = n$, each element of B must occur as an output of f and therefore f is surjective.

$$f \text{ is injective } \Longleftarrow f \text{ is surjective.}$$

Proof by contradiction: In this proof we will begin by assuming that our hypothesis is false. This will lead us to a logical contradiction, forcing us to conclude that our hypothesis is true.

Suppose f is surjective but not injective. Since f is not injective, there must be inputs $x, y \in A$ such that $f(x) = f(y)$. Yet because there are only n inputs to f, this implies that there can be at most $n-1$ distinct outputs of f and thus that at least 1 element of B does not occur as an output of f. However, this contradicts our assumption that f is surjective, and so we must conclude that it is impossible to find a function that is surjective but not injective in this situation.

8. Solve P1.3-2. You need not show any steps.

   Solution:

   1 8 28 56 70 56 28 8 1

9. Solve P1.3-3, part (c).

   Solution:

   $(x+2)^5 = x^5 + \binom{5}{1}x^4 2^1 + \binom{5}{2}x^3 2^2 + \binom{5}{3}x^2 2^3 + \binom{5}{4}x 2^4 + 2^5 =$
   $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

10. Solve P1.3-8.

    Solution:

    $\binom{m+n}{n}$ or $\binom{m+n}{m}$, because

    • From any point, we have two choices: horizontal and vertical. Clearly, we need n vertical lines and m horizontal lines to reach (m, n).

    • Once we have made our choice of n vertical lines, we have also decided the choice of m horizontal lines as these lines are the ones joining the chosen n vertical lines. It can be vice-versa, that is, we choose m horizontal lines which automatically decides the choice of n vertical lines.