CS 19
Winter 2006
Discrete Mathematics

Homework 3 (Grader: Blinn)
Due Feb 1, 2006 by 10:00am

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Please submit your solutions into the box marked "CS 19 Homework In" at the main entrance of Sudikoff.

You must demonstrate how you arrived at your final answers — i.e., you must show your steps — unless the problem statement makes an exception. You must also justify any steps that are not trivial. Simply writing down a final answer *will not earn any credit*. Please think carefully about how you are going to organise your answers *before* you begin writing.

The notation P$i.j$-k refers to Problem k from the list of problems after Section $i.j$ in your textbook. Thus, P1.2-4 refers to Problem 4 on page 17.

1. Solve P1.3-10. [10 points]

2. By rearranging the letters in the word BULB it is possible to form 12 letter strings, as follows:

BULB	BLUB	BUBL	BLBU	BBLU	BBUL
LUBB	LBBU	LBUB	ULBB	UBBL	UBLB

For each of the following words, how many letter strings can you form by rearranging the letters in the word?

2.1. HATCH [5 points]

2.2. UNCOPYRIGHTABLE [5 points]

2.3. SLEEPLESSNESSES [5 points]

3. Solve P1.3-19. [10 points]

4. Solve P1.4-5. [10 points]

5. Figure out the sum of the *odd* binomial coefficients. To be precise, work out the sum

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \binom{n}{i} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

If you are stuck for ideas, first try computing the sum for a few small values of n and see if that gives you a hint. If you are still stuck, look at P1.3-17. [10 points]

[continued on next page]

6. Many equations involving binomial coeffients can be proven true in two ways. An *algebraic proof* simply uses the formula for $\binom{n}{k}$ on both sides of the equation and simplifies, or perhaps it uses other algebraic facts, such as the binomial theorem. A *combinatorial proof* interprets each side as counting a certain set of objects in two different ways. Here is an example.

   Theorem: $\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}$.

   Algebraic Proof: Using the formula for binomial coefficients on the left hand side, we have

   $$\binom{n}{k}\binom{n-k}{j} \;=\; \frac{n!}{k!\cancel{(n-k)!}} \times \frac{\cancel{(n-k)!}}{j!(n-k-j)!} \;=\; \frac{n!}{k!j!(n-k-j)!}\,,$$

   and using it on the right hand side, we have

   $$\binom{n}{j}\binom{n-j}{k} \;=\; \frac{n!}{j!\cancel{(n-j)!}} \times \frac{\cancel{(n-j)!}}{k!(n-j-k)!} \;=\; \frac{n!}{j!k!(n-j-k)!} \;=\; \frac{n!}{k!j!(n-k-j)!}\,.$$

   Thus, we see that the two sides are equal. QED.

   Combinatorial Proof: Let $S = \{1, 2, 3, \ldots, n\}$. The left hand side counts the number of ways of choosing a k-element subset of S followed by a j-element subset of the $n-k$ elements not previously chosen. In other words, it counts the number of pairs (A, B) where $A, B \subseteq S$, $|A| = k$, $|B| = j$ and $A \cap B = \emptyset$. Such a pair (A, B) can also be formed by choosing B first and then choosing A disjoint from B. Counting the pairs this way gives us the right hand side.

   Thus, we see that the two sides are equal. QED.

   For each of the following equations, give an algebraic *as well as* a combinatorial proof.

   6.1. $\displaystyle\sum_{k=0}^{n}\binom{n}{k} \;=\; 2^n.$

   Hint for the algebraic proof: don't expand using factorials; use the binomial theorem. [10 points]

   6.2. $\displaystyle\binom{n}{k}\binom{k}{j} \;=\; \binom{n}{j}\binom{n-j}{k-j}.$

   Hint for the combinatorial proof: consider a subset of a subset. [10 points]

   6.3. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} \;=\; \binom{2n}{n}.$

   Hint for the algebraic proof: use the binomial theorem to analyze the expression $(1+x)^n(1+x)^n$; what is the coefficient of x^n after you multiply out and collect like terms?

   Hint for the combinatorial proof: consider choosing n persons out a group of n men and n women.

   [15 points]

7. Solve P1.4-15. If you don't know `C++` syntax, feel free to write your improved code in the language of your choice. If you know what "pseudocode" means, you may use that too. [10 points]