CS19: Solutions to Homework 4

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1. The base case is obviously true: $1^3 = \frac{1^2(1+1)^2}{4}$.

   By the induction hypothesis, we know that

   $$1^3 + 2^3 \cdots + (n-1)^3 = \frac{(n-1)^2 n^2}{4}.$$

   Now, for the case of n:

   $$(1^3+2^3\cdots+(n-1)^3)+n^3 = \frac{(n-1)^2 n^2}{4}+n^3 = \frac{n^2(n^2-2n+1+4n)}{4} = \frac{n^2(n^2+2n+1)}{4} = \frac{n^2(n+1)^2}{4}.$$

2. **Proof by Induction**:

   The base case $n = j$:

   $$\sum_{i=j}^{j} \binom{i}{j} = \binom{j}{j} = \binom{j+1}{j+1}.$$

   The induction hypothesis for the case $n-1$ is:

   $$\sum_{i=j}^{n-1} \binom{i}{j} = \binom{n}{j+1}.$$

   The case for n is:

   $$\sum_{i=j}^{n} \binom{i}{j} = \sum_{i=j}^{n-1} \binom{i}{j} + \binom{n}{j} = \binom{n}{j+1} + \binom{n}{j} = \binom{n+1}{j+1}.$$

   Combinatorial Proof:

   We know that the number of ways to choose a $(j+1)$-element subset from the $(n+1)$-element set is exactly the right hand side of the equation: $\binom{n+1}{j+1}$.

   Now we partition all these $(j+1)$-element subsets into groups: $G_{j+1}, G_{j+2}, \ldots, G_{n+1}$, where G_x is the set of all $(j+1)$-element subsets having x as the largest integer. (Think why we don't have $G_j, G_{j-1}, \ldots, G_1$.)

What is the size of G_x? Since x is the *largest* number that all the subsets in G_x have, it means we are counting the ways that we can choose $j+1$ items from the set $\{1, 2, \ldots, x\}$ on the condition that x must be chosen (in other words, we are picking j items out of $x-1$ items). So $|G_x| = \binom{x-1}{j}$.

By the sum principle, the total number of subsets, considering all possible values of x, is given by:

$$|G_{j+1}| + |G_{j+2}| + \cdots |G_{n+1}| = \binom{j}{j} + \binom{j+1}{j} + \cdots + \binom{n}{j},$$

which is the left side of the equation. Thus, the two sides are equal. QED.

3. The base case $n = 1$ is easy:

$$\frac{x^3 - 2x^2 + x}{(1-x)^2} = \frac{x(x^2 - 2x + 1)}{x^2 - 2x + 1} = x.$$

Now the induction hypothesis tells us that for the case of $n-1$,

$$\frac{(n-1)x^{n+1} - nx^n + x}{(1-x)^2} = \sum_{i=1}^{n-1} ix^i.$$

For the case of n.

$$\begin{aligned}
\sum_{i=1}^{n} ix^i &= nx^n + \sum_{i=1}^{n-1} ix^i \\
&= \frac{(n-1)x^{n+1} - nx^n + x}{(1-x)^2} + nx^n \\
&= \frac{(n-1)x^{n+1} - nx^n + x + nx^n - 2nx^{n+1} + n^{n+2}}{(1-x)^2} \\
&= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.
\end{aligned}$$

4. The number of the fish in this year is going to be twice of last year, plus the 2000 we dump in. So, we have the equation $M(n) = 2M(n-1) + 2000$.

We can expand the above equation repeatedly:

$$\begin{aligned}
M(n) &= 2M(n-1) + 2000 \\
&= 2(2M(n-2) + 2000) + 2000 \\
&= 2(2(2M(n-3) + 2000) + 2000) + 2000 \\
&= \cdots \\
&= 2^{n-1}M(1) + 2000(1 + 2^1 + 2^2 + \cdots + 2^{n-2}) \\
&= 2^{n-1}M(1) + 2000(2^{n-1} - 1).
\end{aligned}$$

The problem does not tell us what is the number of the fish *in the beginning*, so we just write $M(1)$ as an unknown quantity.

5. By expanding the equation repeatedly:

$$\begin{aligned} T(n) &= rT(n-1) + r^n \\ &= r(rT(n-2) + r^{n-1}) + r^n \\ &= \cdots \\ &= r^n T(0) + nr^n \\ &= (n+1)r^n \end{aligned}$$

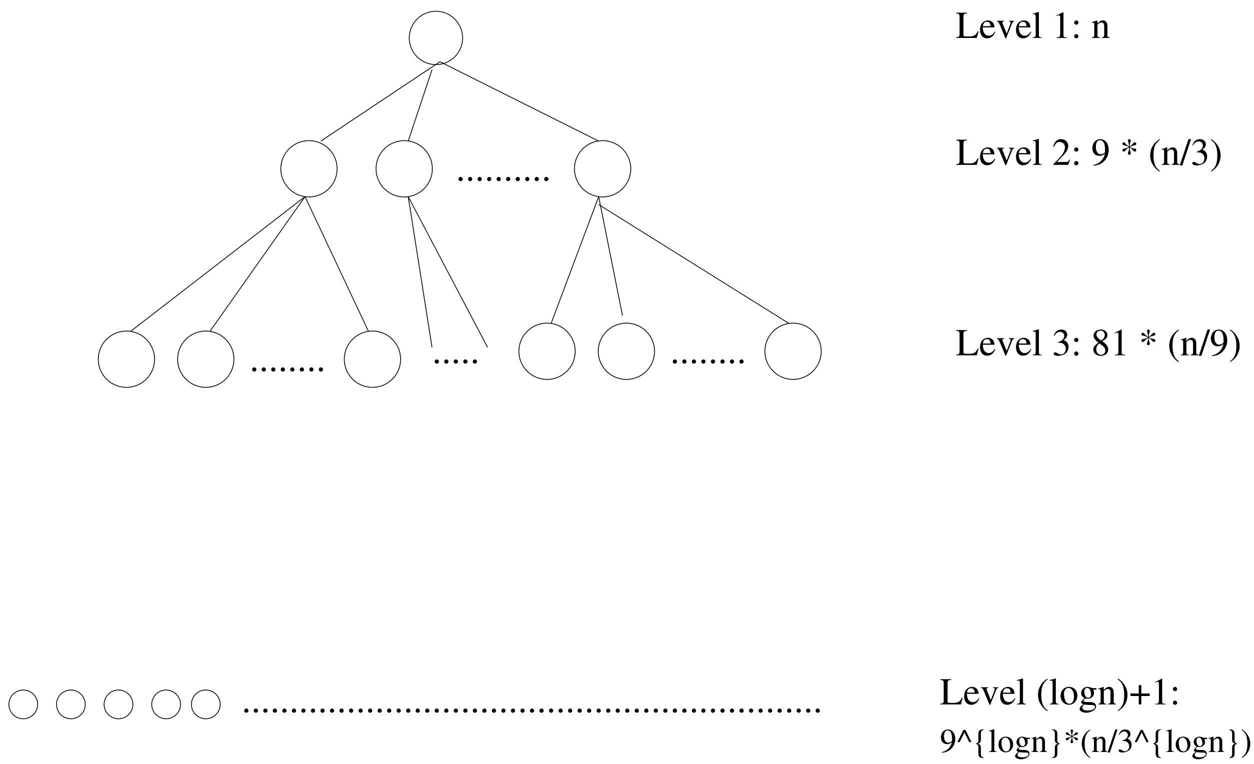
6. Similarly, we expand the equation:

$$\begin{aligned} T(n) &= rT(n-1) + n \\ &= r(rT(n-2) + n - 1) + n \\ &= r(r(rT(n-3) + n - 2) + n - 1) + n \\ &= \cdots \\ &= r^n T(0) + n + (n-1)r + (n-2)r^2 + \cdots + r^{n-1}. \end{aligned}$$

We have to find the summation of $n + (n-1)r + (n-2)r^2 + \cdots + r^{n-1}$, which we denote as S.

$$\begin{aligned} S &= n + (n-1)r + (n-2)r^2 + \cdots + r^{n-1} \\ rS &= \qquad\qquad nr + (n-1)r^2 + \cdots + 2r^{n-1} + r^n \\ (r-1)S &= (r + r^2 + \cdots + r^n) - n \\ (r-1)S &= \frac{r(r^n - 1)}{r-1} - n \\ S &= \frac{r(r^n-1)}{(r-1)^2} - \frac{n}{r-1}. \end{aligned}$$

Finally, we have $T(n) = r^n T(0) + S = r^n + \frac{r(r^n-1)}{(r-1)^2} - \frac{n}{r-1}$.



7. As the above figure shows, the total "work-load" is:

$$
\begin{aligned}
n + 3n + 9n + \cdots 3^{\log_3 n} n &= n(1 + 3 + 9 + \cdots 3^{\log_3 n}) \\
&= n\left(\frac{3^{\log_3 n + 1} - 1}{3 - 1}\right) \\
&= \Theta(n^2).
\end{aligned}
$$

8. As in the last problem, we can sum up the work-load as

$$
\begin{aligned}
n + \frac{n}{2} + \frac{n}{4} \cdots + \frac{n}{2^{\log_2 n}} &= n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{\log_2 n}}\right) \\
&= n \cdot \frac{1 - (1/2)^{2^{\log_2 n + 1}}}{1 - 1/2} \\
&= \Theta(n).
\end{aligned}
$$