CS19: Solutions to Homework 5

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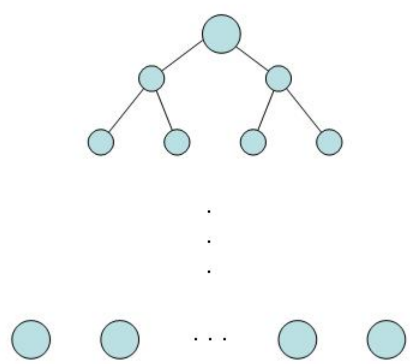
The notation P$i.j$-k refers to Problem k from the list of problems after Section $i.j$ in your textbook. Thus, P1.2-4 refers to Problem 4 on page 17.

1. Solve P4.3-14, parts (b) and (c). Please draw out your recursion tree neatly, following the same scheme as the examples in the book.

   Solution:

   (b)

Number of problems	Problem size	Work per problem	Work per level
1	*n*	*n*	*n*
2	*n-1*	*n-1*	*2(n-1)*
2^2	*n-2*	*n-2*	2^2 *(n-2)*
2^{n-2}	*n-(n-2)*	*n-(n-2)*	2^{n-2} *(2)*
2^{n-1}	*n-(n-1)*	*n-(n-1)*	2^{n-1} *T(1)* = 2^{n-1}



So,

$$\begin{aligned}
T(n) &= \sum_{i=0}^{n-1} 2^i(n-i) \\
&= n\sum_{i=0}^{n-i} 2^i - \sum_{i=0}^{n-1} i2^i \\
&= n\left(\frac{2^{n-1}}{2-1}\right) - \frac{n2^{n+1} - n2^n - 2^{n+1} + 2}{(2-1)^2} \qquad \left[\text{using the result } \textstyle\sum_{i=0}^{n-1} ib^i = \frac{nb^{n+1} - nb^n - b^{n+1} + b}{(b-1)^2}\right] \\
&= n2^n - n - n2^{n+1} + n2^n + 2^{n+1} - 2 \\
&= 2^{n+1} - n - 2 \\
&= O(2^n)\,.
\end{aligned}$$

(c)

Number of problems	Problem size	Work per problem	Work per level
1	2^{2^i}	1	1
1	$2^{2^{i-1}}$	1	1
1	$2^{2^{i-2}}$	1	1
	.		
	.		
1	$2^{2^{i-(i-1)}}$	1	1
1	$2^{2^{i-i}}=2^0=1$	1 $T(1) = 1$	1 $T(1) = 1$

So, $T(n) \;=\; \sum_{j=0}^{i} 1 \;=\; i+1$

Now we use the fact that if $n = 2^{2^i}$, taking log to the base 2 on both sides, we get, $\log_2 n = 2^i$ and taking another log to the base 2 on both sides, we get $\log_2 \log_2 n = i$.

Thus, $T(n) \;=\; i+1 \;=\; \log_2 \log_2 n + 1 \;=\; \Theta(\log \log n)$.

2. Solve P4.4-1 (all five parts). Do not draw recursion trees; just use the master theorem. You must show the steps that led you to one of the three cases of the master theorem.

   Solution:

   (a) $a = 8, b = 2, c = 1$
   $\log_2 8 = 3 > c = 1$. So, we have,
   Case 3: $\Theta(n^3)$.

   (b) $a = 8, b = 2, c = 3$
   $\log_2 8 = 3 = c$. So. we have,
   Case 2: $\Theta(n^3 \log n)$.

   (c) $a = 3, b = 2, c = 1$
   $\log_2 3 > c = 1$. So, we have,
   Case 3: $\Theta(n^{\log_2 3})$.

   (d) $a = 1, b = 4, c = 0$
   $\log_4 1 = c = 0$. So, we have,
   Case 2: $\Theta(\log n)$.

   (e) $a = 3, b = 3, c = 2$
   $\log_3 3 = 1 < c = 2$. So, we have,
   Case 1: $\Theta(n^2)$.

3. Solve P4.4-2, P4.4-3, P4.4-4 and P4.4-5. Use the general version of the master theorem that handles ceilings.

   Solution:

   P4.4-2 Since $\sqrt{n+3} = \Theta(n^{1/2})$, and $\log_2 3 > 1/2$, we have according to case 3 of the master theorem that $T(n) = \Theta(n^{\log_2 3})$.

P4.4-3 Since $\sqrt{n^3+3} = \Theta(n^{3/2})$ and $\log_2 3 > 3/2$, we have according to case 3 of the master theorem that $T(n) = \Theta(n^{\log_2 3})$.

P4.4-4 Since $\sqrt{n^4+3} = \Theta(n^{4/2}) = \Theta(n^2)$, and $\log_2 3 < 2$, we have according to case 1 of the master theorem that $T(n) = \Theta(n^2)$.

P4.4-5 Since $\sqrt{n^2+3} = \Theta(n^{2/2}) = \Theta(n)$, and $\log_2 2 = 1$, we have according to case 2 of the master theorem that $T(n) = \Theta(n \log_2 n)$.

4. Solve P4.5-3. Mimic the style of the induction proofs in exercises 4.5-{1,2,3}.

   Solution: We are given that $T(n) \leq 2T(n/3) + c \log_3 n$. We may assume n is a power of three. We want to show that there is a n_0 and a $k > 0$ so that $T(n) \leq kn \log_3 n$ for $n \geq n_0$. Since $\log_3 1 = 0$, n_0 must be at least 2. But we are assuming n is a power of 3, so we may as well try assuming $n_0 = 3$. Since we want $T(3) \leq k3 \log_3 3 = 3k$, we must have $k \geq T(3)/3$. Suppose inductively that $n > 3$ and for all $m < n$, $T(m) \leq km \log_3 m$. Then

$$\begin{aligned}
T(n) &\leq 2\left(\frac{kn}{3} \log_3\left(\frac{n}{3}\right)\right) + c \log_3\left(\frac{n}{3}\right) \\
&= \frac{2kn}{3} \log_3 n - \frac{2kn}{3} + c \log_3 n - c \\
&= \left(\frac{2kn}{3} + c\right) \log_3 n - \frac{2kn}{3} - c \\
&= (kn - \frac{1}{3}kn + c) \log_3 n - \frac{2kn}{3} - c
\end{aligned}$$

   We want this to be no more than $kn \log_3 n$. Since $n > 3$, we have $\frac{1}{3}kn > k$, so as long as we choose $k > c$, we have $T(n) = kn \log_3 n$. Thus by the principle of mathematical induction, so long as k is at least c and at least $T(3)/3$, we have $T(n) = kn \log_3 n$ for all $n \geq 3$ with n being a power of 3. If we replace the 2 by 3, the same kind of argument works, but it is more delicate. (We lose the helpful $\frac{1}{3}kn$ and have to play a $-kn$ off against a $c \log_3 n$.) Since $\log_b n = \Theta(\log_3 n)$, changing the base of the logarithm just amounts to changing the constant c, so the same proof works.

5. Supposing we replace "$\leq$" with "$=$" in the recurrence in problem 4. Now, does the $O(n \log_3 n)$ bound of problem 4 become a $\Theta(n \log_3 n)$ bound?. Solve this problem and *explain your answer*; i.e., if you think that the answer is "no," then state and prove the correct big-Θ bound. Note that the master theorem does not apply in this case, so you will not be able to use it.

   Solution: No, the $O(n \log_3 n)$ bound in the previous problem is not a $\Theta(n \log_3 n)$ bound. To see why, note that $\log_3 n = O(n^{0.1})$. The complexity of the original recurrence, $T(n) = 2T(n/3) + c \log_3 n$, must then be less than or equal to the complexity of a new recurrence, $S(n) = 2S(n/3) + cn^{0.1}$ (meaning $T(n) = O(S(n))$). Note that we can apply the master theorem to $S(n)$. Since $\log_3 2 > 0.1$, case 3 of the master theorem tells us that $S(n) = O(n^{\log_3 2})$. Because we have obtained a better upper bound, the $O(n \log_3 n)$ bound in the previous problem can not be a $\Theta(n \log_3 n)$ bound. In fact you can prove by induction, or with a recursion tree, that $T(n) = \Theta(n^{\log_3 2})$ (or equivalently, that $T(n) = \Theta(2^{\log_3 n})$).

6. Let α, β and c be positive real constants with $\alpha + \beta < 1$. Suppose $T(n)$ is a sequence defined on the integers that satisfies the inequality

$$T(n) \;\leq\; T(\lceil \alpha n \rceil) + T(\lceil \beta n \rceil) + cn\,.$$

   Give a careful proof, using induction and the precise definition of big-O, that $T(n) = O(n)$.

Solution: We shall prove that there exists a positive constant k such that $T(n) \le kn$ for every integer $n \ge 1$. We shall prove this using strong induction. Choose n_0 and k so that they satisfy

$$\begin{aligned} n_0 &\ge \frac{4}{1-\alpha-\beta}\,, && (1) \\ k &\ge \max_{1 \le i \le n_0} \left\{ \frac{T(i)}{i} \right\}\,, && (2) \\ k &\ge \frac{2c}{1-\alpha-\beta}\,. && (3) \end{aligned}$$

Note that the quantity $(1-\alpha-\beta)$, which appears twice above and several more times below, is positive because we have been given $\alpha + \beta < 1$. This fact is used tacitly in what follows. First, we make (and prove) two important claims:

Claim 1: For $n \ge n_0$, we have $\lceil \alpha n \rceil < n$ and $\lceil \beta n \rceil < n$.

Claim 2: For $n \ge n_0$, we have $(1-\alpha-\beta)n - 2 \ge \dfrac{(1-\alpha-\beta)n}{2}$.

Proof of Claim 1: Using (1), $(1-\alpha-\beta)n \ge (1-\alpha-\beta)n_0 \ge 4$. We can rewrite that as $\alpha n \le n - \beta n - 4$. Now, using this, $\lceil \alpha n \rceil < \alpha n + 1 \le n - \beta n - 4 + 1 < n$. Similarly, $\lceil \beta n \rceil < n$.

Proof of Claim 2: Using (1), $\frac{2}{n} \le \frac{2}{n_0} \le 2(\frac{1-\alpha-\beta}{4}) = \frac{1-\alpha-\beta}{2}$. Thus,

$$\frac{(1-\alpha-\beta)n - 2}{n} = (1-\alpha-\beta) - \frac{2}{n} \ge (1-\alpha-\beta) - \frac{1-\alpha-\beta}{2} = \frac{1-\alpha-\beta}{2}\,.$$

Multiplying out by n gives us the inequality we claimed.

Let us return to the problem at hand. As our base cases, we first establish $T(n) \le kn$ for $1 \le n \le n_0$. This is easy: it follows directly from (2).

For the inductive step, suppose $n > n_0$ and suppose we have shown $T(i) \le ki$ for all positive integers $i < n$. We shall show that $T(n) \le kn$ as well. Claim 1 tell us that $\lceil \alpha n \rceil < n$ and $\lceil \beta n \rceil < n$, so the inductive hypothesis applies and we get

$$\begin{aligned} T(n) &\le T(\lceil \alpha n \rceil) + T(\lceil \beta n \rceil) + cn \\ &\le k\lceil \alpha n \rceil + k\lceil \beta n \rceil + cn \\ &< k(\alpha n + 1) + k(\beta n + 1) + cn \\ &= kn - k((1-\alpha-\beta)n - 2) + cn\,. \end{aligned}$$

Now, using Claim 2,* we obtain

$$T(n) \le kn - k\frac{(1-\alpha-\beta)n}{2} + cn\,.$$

Next, using (3),† we obtain

$$T(n) \le kn - \frac{2c}{1-\alpha-\beta}\frac{(1-\alpha-\beta)n}{2} + cn = kn - cn + cn = kn\,.$$

This completes the inductive step.

*Do you see why we made such an outlandish looking claim?

†Do you now see why we wanted k to satisfy inequality (3)?