CS19: Solutions to Homework 6

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You must demonstrate how you arrived at your final answers — i.e., you must show your steps — unless the problem statement makes an exception. You must also justify any steps that are not trivial. Simply writing down a final answer *will not earn any credit.* Please think carefully about how you are going to organise your answers *before* you begin writing.

The notation P*i.j-k* refers to Problem k from the list of problems after Section $i.j$ in your textbook. Thus, P1.2-4 refers to Problem 4 on page 17.

1. Solve both parts of P5.1-6.

   Solution:

   (a)

Sample Space (**Coin$_1$,Coin$_2$**)	**Probability Weight**
(P, P)	2/12
(P, N)	2/12
(P, D)	2/12
(N, P)	2/12
(N, D)	1/12
(D, P)	2/12
(D, N)	1/12

   (b) Eleven must be one penny and one dime or one dime one penny. From above, the probability of (P, D) or (D, P) is $(2/12 + 2/12) = 1/3$.

2. Solve P5.1-11.

   Solution: A student who gets 80 or higher, may get 100, 90, or 80. There are 2^{10} patterns of right and wrong answers. There is one way to get 100. There are $\binom{10}{1} = 10$ ways to get 90. There are $\binom{10}{2} = 45$ ways to get 80. Thus, the probability of scoring an 80 or above is

   $$\frac{1 + 10 + 45}{2^{10}} = \frac{56}{1024} = \frac{7}{128} = 0.0546875 .$$

   The probability of a score of 70 or lower is $1 - 7/128 = 121/128$, which is 0.9453125.

3. Poker is a rather mathematical card game with many variations, but all based on the basic principle that certain hands "beat" (i.e., have higher value than) certain others. A hand consists of five distinct cards chosen from a standard deck of 52 cards. There are nine special hands and they are given specific names, as follows.

   A *royal flush* consists of the Ace, King, Queen, Jack and 10, all of the same suit. A *straight flush* is any five-card sequence within a suit, except for the one beginning with the Ace (that would make it royal), for instance, Jack, 10, 9, 8, 7. A *straight* is any five-card sequence with not all cards of the same suit and a *flush* is a set of five cards of the same suit but not in sequence.

A *four-of-a-kind* is hand with four cards of the same value (e.g. with four 9s). A *three-of-a-kind* has three cards of the same value and two other cards with two other different values; had these two other cards had the same value that would give us a *full house*; thus, three 5s, a King and a 9 give us a three-of-a-kind whereas three 5s and two 10s give us a full house. A *two pair* contains two different equal-value pairs and an unrelated fifth card, e.g. two 7s, two Kings, and an Ace. Finally, a *pair* has just one equal-value pair and three other unrelated cards.

(a) Suppose you shuffle a deck of 52 cards and then draw five cards at random. What is the probability of getting each of the nine special hands (royal flush, straight flush, straight, flush, four-of-a-kind, three-of-a-kind, full house, two pair and pair)?

(b) A sensible design of the rules of poker would ensure that if you have been lucky enough to draw an "unlikely" hand, then your hand should beat that of an opponent who has drawn a more "mundane" hand. In fact, poker *was* sensibly designed. By arranging the hands from least likely to most likely, figure out the "pecking order" of these special hands in poker. You might want to use a calculator unless you're a whiz with numbers!

Solution: A sample space for our experiment (of drawing five random cards from a deck) is the family of all 5-card subsets (i.e., "hands") of the set of all 52 cards. The hands are equally likely* and so the probability measure is uniform. The size of the sample space is $\binom{52}{5}$. Therefore, to find the probability of an event like "the hand is a straight flush" we simply count the number of hands that are straight flushes and divide by $\binom{52}{5}$.

- For a royal flush, we have only 4 choices: we choose the suit, and the values of the cards are fixed: $\{A, K, Q, J, 10\}$. Thus there are exactly 4 hands which are royal flushes.
- For a straight flush, we get to choose the suit (4 choices) and the value of the highest card (8 choices: anything from K to 6). Therefore, there are $4 \times 8 = 32$ total straight flushes.
- For a straight, we get to choose the value of the top card (9 choices: anything from A to 6) and this fixes the values of the other four cards. We still have many choices for the suits: 4 choices for each card, giving $4^5 = 1024$ choices, except that four of these choices are "bad" because they put all cards in the same suit. So, overall, we have 9 choices for the top card's value times $1024 - 4 = 1020$ choices for the suits, giving $9 \times 1020 = 9180$ straights.
- For a flush, having chosen the suit (4 choices), we are left with $\binom{13}{5} = 1287$ ways to pick the values of the five cards, except that 9 of these choices are "bad" because the values are in sequence. Overall, this gives us $4 \times (1287 - 9) = 5112$ flushes.
- The four equal-valued cards in a four-of-a-kind can be chosen in 13 ways (we can only choose the value). The fifth card can then be any of the $52 - 4 = 48$ cards remaining in the deck. This gives us $13 \times 48 = 624$ such hands.
- In a three-of-a-kind, we still have 13 ways to choose the value of the triplet, but now we also have $\binom{4}{3} = 4$ ways to choose which three suits these cards will be in. The remaining two cards must have two *distinct* values out of the 12 other values — $\binom{12}{2} = 66$ choices — and for each choice of values we have $4^2 = 16$ ways to assign them suits. Putting it all together, the number of three-of-a-kind hands is $13 \times 4 \times 66 \times 16 = 54912$.†
- For a full house, there are 13 ways to choose the value of the triplet and then 12 ways to choose the value of the doublet. Note that the number of ways of choosing these two values together is

*If it bothers you that we are using unordered subsets instead of ordered lists of length 5, then just note that each subset corresponds to exactly $5! = 120$ lists. Unlike in the case of rolling dice, where both dice might show the same number, there is no chance here of two cards in a hand being exactly the same.

†Note that we didn't need to worry about "accidentally" hitting a flush or a straight, since the presence of three equal-valued cards already prevented that.

not $\binom{13}{2}$ because order matters: we care which value is for the triplet and which for the doublet. After this, there are $\binom{4}{3} = 4$ ways to pick the suits of the cards in the triplet and $\binom{4}{2} = 6$ ways to do it for the doublet. Overall, this give us $13 \times 12 \times 4 \times 6 = 3744$ full houses.

- Counting two pair hands can be somewhat tricky: *pay attention!* There are $\binom{13}{2} = 78$ ways to pick the values of the pairs. Having picked two values, we can choose the suits of the lower valued pair in $\binom{4}{2} = 6$ ways and those of the higher valued pair in $\binom{4}{2} = 6$ ways. The fifth card is forbidden from having either of these two values, so there are $4 \times (13 - 2) = 44$ ways of picking it. Thus, there are $78 \times 6 \times 6 \times 44 = 123552$ two pairs.
- For a pair, we have 13 choices for the paired value and $\binom{4}{2} = 6$ ways to pick suits for the paired cards. The remaining three cards must have three other distinct values, giving us $\binom{12}{3} = 220$ choices. We can then choose a 3-tuple of suits ($4^3 = 64$ choices) and then sort these three cards by value and match suits to values (there's only one way to do this). Overall, we have $13 \times 6 \times 220 \times 64 = 1098240$ pairs.

As mentioned earlier, the probability values are obtained by dividing these numbers by $\binom{52}{5} = 2598960$. For completeness's sake, here are the values. They are sorted, so the table below also reveals the pecking order of hands in poker.

Hand type	Number	Probability
Royal flush	4	.0000015
Straight flush	32	.0000123
Four-of-a-kind	624	.0002401
Full house	3744	.0014406
Flush	5112	.0019669
Straight	9180	.0035322
Three-of-a-kind	54912	.0211285
Two pair	123552	.0475390
Pair	1098240	.4225690

Note that in many games of Poker, an Ace may be given a value of either above a King or below a Two. In choosing his or her hand, a player may set the value of the Ace to achieve the best hand possible. Under these rules, the hand (A 2 3 4 5) is also considered a straight. Taking this into account, the odds for achieving a straight or straight flush change slightly. Either the solution above or the modified solution below will be given credit.

- For a straight flush, we get to choose the suit (4 choices) and the value of the highest card (9 choices: anything from K to 5). Therefore, there are $4 \times 9 = 36$ total straight flushes.
- For a straight, we get to choose the value of the top card (10 choices: anything from A to 5) and this fixes the values of the other four cards. We still have many choices for the suits: 4 choices for each card, giving $4^5 = 1024$ choices, except that four of these choices are "bad" because they put all cards in the same suit. So, overall, we have 9 choices for the top card's value times $1024 - 4 = 1020$ choices for the suits, giving $10 \times 1020 = 10200$ straights.

The pecking order of hands remains the same, although with slightly different odds.

Hand type	Number	Probability
Royal flush	4	.0000015
Straight flush	36	.0000139
Four-of-a-kind	624	.0002401
Full house	3744	.0014406
Flush	5112	.0019669
Straight	10200	.0039246
Three-of-a-kind	54912	.0211285
Two pair	123552	.0475390
Pair	1098240	.4225690

4. Solve P5.2-10. You will want to use the principle of inclusion and exclusion and your final answer will be a formula using summation (i.e., Σ) notation. It is not possible to simplify the summation, so leave it at that.

   Let E_i be the event that location i is empty. We are interested in $1 - P(E_1 \cup E_2 \cup \ldots \cup E_k)$. By Equation 5.6 in the textbook,

   $$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k : \\ 1 \le i_1 \le i_2 \le \ldots \le i_k \le n}} P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) .$$

   We can see that $P(E_1) = \frac{(k-1)^n}{k^n}$, $P(E_1 \cap E_2) = \frac{(k-2)^n}{k^n}$, and in general:

   $$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = \frac{(k-j)^n}{k^n}.$$

   There are $\binom{k}{j}$ subsets of size j, so we have $1 - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \frac{(k-j)^n}{k^n}$ for the probability that each location gets one key. Another way, using the principle of inclusion and exclusion for counting: The problem can be transformed to the problem of how many onto mappings are there from an n-element set to a k-element set. Thus, we have

   $$\frac{k^n - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^n}{k^n} = \frac{\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} (k-j)^n}{k^n}$$

   In deriving this equation, we have used the fact that total number of ways of mapping the n-element set to a k-element set is k^n (each of the n-element can be mapped to one of the k-elements).

5. Find the number of integers between 1 and 10,000 (inclusive) that are not divisible by 4, nor by 5, nor by 6.

   Solution: Let $A = \{1, 2, 3, \ldots, 10000\}$ and for an arbitrary integer k, let $A_k = \{x \in A : x \text{ is divisible by } k\}$, i.e., A_k is the set of those integers from 1 through 10000 that are divisible by k. The problem asks for the size of the set $A - (A_4 \cup A_5 \cup A_6)$. Since $(A_4 \cup A_5 \cup A_6)$ is a subset of A, the size of the difference of the two sets is the difference of their sizes; thus, the answer we seek is

   $$|A - (A_4 \cup A_5 \cup A_6)| = |A| - |A_4 \cup A_5 \cup A_6| = 10000 - |A_4 \cup A_5 \cup A_6| .$$

   Now, by the principle of inclusion and exclusion,

   $$|A_4 \cup A_5 \cup A_6| = |A_4| + |A_5| + |A_6| - |A_4 \cap A_5| - |A_4 \cap A_6| - |A_5 \cap A_6| + |A_4 \cap A_5 \cap A_6| .$$

It is easy to figure out $|A_4|$. Every fourth integer is divisible by 4, so the number of integers from 1 to 10000 that are divisible by 4 is simply $\lfloor 10000/4 \rfloor = 2500$. Thus, $|A_4| = 2500$. Similarly, $|A_5| = \lfloor 10000/5 \rfloor = 2000$ and $|A_6| = \lfloor 10000/6 \rfloor = 1666$.

What about $A_4 \cap A_5$? Some experimentation should convince you that an integer is divisible by both 4 and 5 if and only if it is divisible by 20, i.e., $A_4 \cap A_5 = A_{20}$ (think of how you might prove this formally; we'll see a proof later in the course). Similarly, $A_5 \cap A_6 = A_{30}$. But $A_4 \cap A_6 = A_{12}$ and *not* A_{24}! (why?) and $A_4 \cap A_5 \cap A_6 = A_{60}$.[‡] Using all of this we get

$$\begin{aligned} |A_4 \cup A_5 \cup A_6| &= |A_4| + |A_5| + |A_6| - |A_{20}| - |A_{12}| - |A_{30}| + |A_{60}| \\ &= 2500 + 2000 + 1666 - 500 - 833 - 333 + 166 \\ &= 4666\,. \end{aligned}$$

This is the number of integers in A that *are* divisible by one of $4, 5, 6$. As we mentioned before, the answer we seek is 10000 minus this, i.e., 5334.

6. Solve P5.3-2.

   Solution:

   A = event that two flips in a row are heads

   B = event that there is an even number of heads

   $A \cap B$ = the event that two flips in a row are heads and the remaining flip is a tail Now, $P(A \cap B) = 2/2^3 = 1/4$ as HHT and THH are the only possibilities. Also, $P(B) = 1/2$ (HHT, HTH, THH, TTT). So, $P(A|B) = \frac{1/4}{1/2} = 1/2$ Thus, the probability that two flips in a row are heads given that there is an even number of heads is 1/2.

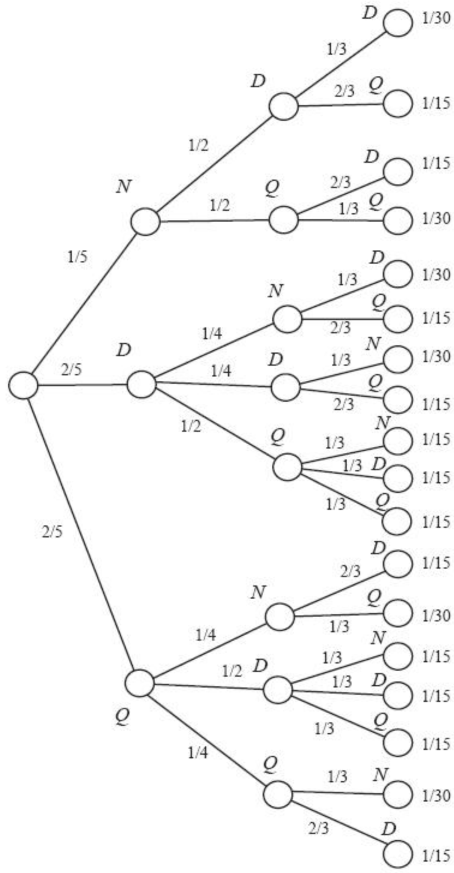
   The event of two flips of heads in a row is the event (HHT, HHH, THH) which has probability 3/8. Thus by the definition of independence, the two events are not independent.

   Another approach to the problem would be to note that the probability of an even number of heads is 1/2, and $1/2 \times 3/8 = 3/16$ while the probability of an even number of heads and two heads in a row is the probability of the event (HHT,THH) which is 1/4. Thus by Theorem 5.5, the events are not independent.

7. Solve P5.3-7.

   The tree diagram is given below. Adding the probabilities adjacent to the paths that end in an N node gives 1/5. The probability of a quarter on the last draw is similarly 2/5, while the probability of a quarter on the first and last draw is 1/10. Therefore the probability that the first coin is a quarter, given that the last coin is a quarter is $\frac{1/10}{2/5} = 1/4$.

[‡] Hopefully, some of you have discovered the general rule: a number is divisible by all of $k_1, k_2, \ldots, k_t$ if and only if it is divisible by the *least common multiple* (lcm) of $k_1, k_2, \ldots, k_t$. If you read Chapter 2 of your textbook, you can learn how to prove this.



8. Solve P5.3-8.

   The number of ways that a bridge hand has four aces is $\binom{48}{9}$ (because having 4 aces in our hand leaves us with 9 cards to choose from the remaining 48 in the deck). The number of ways that a bridge hand has one ace is $\binom{52}{13} - \binom{48}{13}$ ($\binom{52}{13}$ is the total no. of ways a bridge hand can be chosen, and $\binom{48}{13}$ is the no. of ways a bridge hand can be chosen without any ace. So, $\binom{52}{13} - \binom{48}{13}$ gives us the no. of ways a bridge hand can contain at least one ace). Thus, the probability that a bridge hand (which is 13 cards, chosen from an ordinary deck) has four aces, given that it has (at least) one ace is $\frac{\binom{48}{9}}{\binom{52}{13}-\binom{48}{13}} = 0.000947$.

   The number of ways that a bridge hand has four aces is $\binom{48}{9}$. The number of ways that a bridge hand has the ace of spades is $\binom{51}{12}$. Thus, the probability that a bridge hand (which is 13 cards, chosen from an ordinary deck) has four aces, given that it has the ace of spades is $\frac{\binom{48}{9}}{\binom{51}{12}} = 0.002841$.

   Since $0.000947 < 0.002841$., the probability that a bridge hand has four aces given that it has the ace of spades is larger.

9. This problem should convince you not to trust vaguely formed "intuitions" about probability, but instead to carefully work out the numbers using the proper definitions and theorems from probability theory. It is the famous (some would say infamous) Monty Hall problem, which gets its name from the TV game show *Let's Make A Deal*, hosted by Monty Hall.

   You are asked to select one closed door of three, behind one of which there is a prize. The other two doors hide nothing. Once you have made your selection, Monty Hall opens one of the remaining doors, revealing that it does not contain the prize. He then asks you if you would like to switch your selection to the other unopened door, or stay with your original choice. The problem: should you switch?

Work this out meticulously. Carefully define a sample space, define any necessary events and then work out two probabilities:

(a) The probability that you win the prize if you don't switch.

(b) The probability that you win the prize if you do switch.

Do you find the answers intuitive? (There is no incorrect answer to that question!) If not, the lesson you have learnt is that you need to wait until your intuition has matured before trusting it.

Solution: The key is to recognize the difference betwen the following two situations:

(1) The door we pick initially is the prize door.

(2) The door we pick initially is not the prize door.

In situation (1), we will definitely lose if we switch. In situation (2), Monty Hall has no choice about which door he is to open: he *must* open the only door which is different from our pick and from the prize door. Therefore, we will definitely win if we switch. Therefore, all we have to do is work out the probabilities of being in situations (1) and (2).

For this, we can use a sample space consiting of ordered pairs (i, j) where i is the number of the door we pick and j is the number of the prize door; $1 \leq i, j \leq 3$. Since no door is special, all of these $3 \times 3 = 9$ situations are equally likely. By definition, we are in situation (1) in three out the nine cases: $(1, 1)$, $(2, 2)$, and $(3, 3)$. Therefore situation (1) occurs with probability $3/9 = 1/3$ and situation (2) with probability $1 - 1/3 = 2/3$. In other words, the probability that we win if we switch is $2/3$.

If we don't switch, the analysis is exactly as above except that we win in situation (1), i.e., with probability $1/3$ and lose in situation (2), i.e., with probability $2/3$.

Thus, we should most definitely switch. It doubles our winning chances.