CS19: Solutions to Homework 8

Written by Chien-Chung Huang

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1. We denote the number of vertices in each component (tree) as u_i and obviously $\sum_{i=1}^{c} u_i = v$. By Theorem 6.5, the number of edges in each tree is $u_i - 1$. Hence, $e = \sum_{i=1}^{c}(u_i - 1) = v - c$

2. No. We prove by contradiction. Since the connected graph is not a tree, there exists a cycle C in it. We will use the following claim.

   Claim 1 *Given a connected graph G containing a cycle C, if we remove any edge $\{a, b\}$ from C, then the remaining graph is still connected.*

   Proof: After removing $\{a, b\}$ from the cycle C, we denote the new graph as G'. To show G' is connected, we recall the definition : a graph is "connected" if for each pair of vertices, there is a path between them. We now show this definition still holds in G'.

   - If between a pair of vertices (x, y), there exists in the original graph G a path which does not use the edge $\{a, b\}$, then removing $\{a, b\}$ definitely will not influence this existing path.
   - If between a pair of vertices (x, y), all paths in the original graph G use the edge $\{a, b\}$, we pick any one of them and denote it as P. We observe that P is composed of the path $x \to a$, the edge $\{a, b\}$ and $b \to y$. Now, we remove $\{a, b\}$ from the path and replace it with the remaining edges in C. This new walk $x \to a$, $C - \{a, b\}$ and $b \to y$ is a walk connecting x and y without using the edge $\{a, b\}$. Hence, we claim that in G', there is still a walk (and therefore, a path) between x and y.

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   By the claim, we can remove an edge from C and the component is still connected. At this point, we have v vertices and $v - 2$ edges. We claim that at least one vertex will have only degree 1, otherwise $\sum_{i=1}^{v} d_i \geq 2v > 2(v - 2)$, which contradicts Theorem 6.2. Now, we remove this vertex of degree 1 with its incident edge. We will have a graph of $v - 1$ vertices and $v - 3$ edges and still connected. By repeating the argument repeatedly, we will reach the stage that we have 2 vertices and 0 edges and the graph still connected. A contradiction.

3. No. If the graph does not contain cycles but is still not a tree, then by definition, it can only be that the graph is composed of more than one tree (since there is no cycle). From Question 1, we know that $v = e + c = (v-1) + c$. Then it means c, the number of trees, is 1. And we have the contradiction.

4. We claim that G must be a tree. Suppose not. Then we can have a connected graph containing a cycle C and removing any edge will leave the graph disconnected. However, if we remove an edge from the cycle C, the graph is still connected. Thus, we have a contradiction.

5. This problem can be proved by induction. But here we give a combinatorial counting argument.

   Given any number k. We separate the number of vertices into two groups V_A and V_B. The former contains the first k vertices and the latter the last $n-k$ vertices. The left hand side of the equation $\sum_{i=1}^{k} d_i$ is essentially the sum of all the degrees of vertices in V_A. This is the same as counting totally how many edges are incident to these vertices. For these edges, there are two possibilities:

   - They connect the vertices in V_A.
   - They connect the vertices from V_A to V_B.

   For the first case, we know there are at most $\frac{|V_A||V_A-1|}{2}$ of them. Hence, they can contribute at most $|V_A||V_A-1| = k(k-1)$ to the total degrees of vertices in V_A.

   For the second case, consider any node v in V_B. If v's degree d_v is smaller than k, then it can be connected to at most d_v of them. On the other hand, if d_v is greater than or equal to k, still it can be incident to at most k of them. Thus, we know the total degree contribution can be at most $\sum_{i=k+1}^{n} \min\{k, d_i\}$.

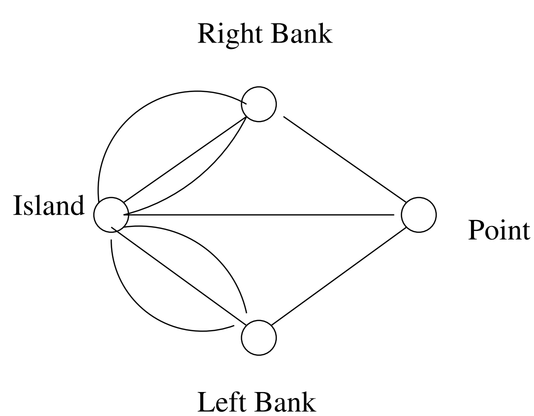
   Combining the two cases, we have the right hand side of the inequality.

6. For (a) and (d), we don't have an Eulerian trial, as there are more than two vertices with odd degrees.

   For (b), the vertex sequence is 123451245. For (c), the vertex sequence is 12541532

7. Now we can have an Eulerian trail. (The island and the point both have odd degrees. The right bank and the left bank have even degrees). The starting place and ending place are going to be the island and the point.

Konigsberg (nowadays Kaliningrad)



8. We give an example in the four dimension cube Q_4. Given any point p, say $(1, 0, 0, 0)$, which other points can be its neighbor?

$$(0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1),$$

   Do you notice that every neighbor differs from p at exactly one coordinate? In other words, the "degree" of any point in Q_n is exactly the number of its dimension. By the definition of Eulerian tour, we must have n is even.