CS 19: Discrete Mathematics

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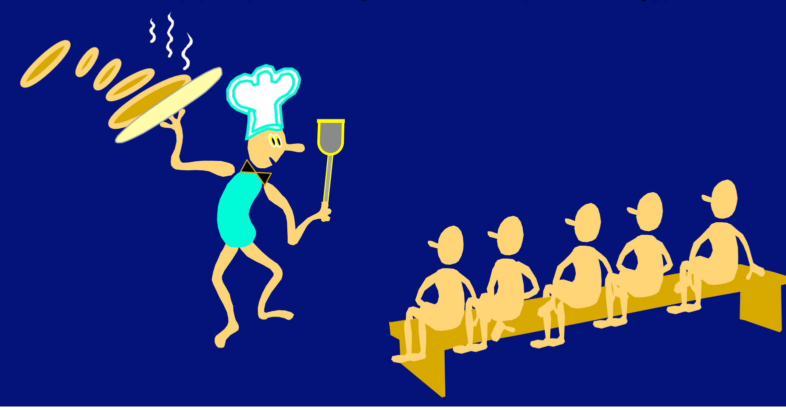
Getting into the mood

- Need to start by learning/reviewing some basic math notation and concepts. This can seem dry.
- Before getting to the juicy fruit, need to deal with the dry skin.
- But why not taste a slice of the fruit first?

Acknowledgment: Today's slice of fruit comes courtesy of Professor Anupam Gupta, CMU. These slides are from his Fall 2005 course "Great Ideas in Theoretical Computer Science".

Pancakes With A Problem!

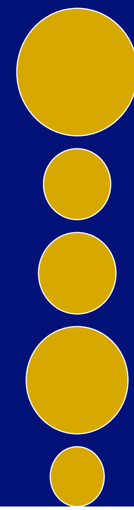


The chef at our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes.

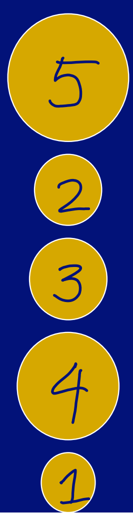
Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom)

I do this by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary.

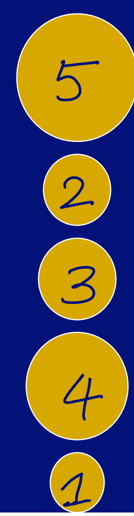
Developing A Notation:
Turning pancakes into numbers



Developing A Notation:
Turning pancakes into numbers



Developing A Notation:
Turning pancakes into numbers



5

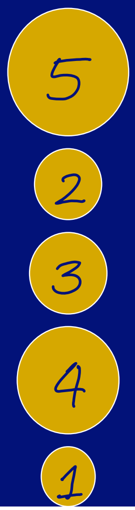
2

3

4

1

Developing A Notation: Turning pancakes into numbers

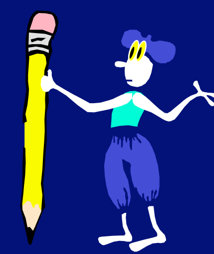


→

5
2
3
4
1

How do we sort this stack? How many flips do we need?

5
2
3
4
1



4 Flips Are Sufficient

5	1	2	4	1
2	4	3	3	2
3	3	4	2	3
4	2	1	1	4
1	5	5	5	5

Algebraic Representation

X = The smallest number of flips required to sort: 5 2 3 4 1

? ≤ X ≤ ?

Lower Bound

Upper Bound

Algebraic Representation

X = The smallest number of flips required to sort: 5 2 3 4 1

Lower Bound

$? \le X \le 4$

Upper Bound

4 Flips are necessary in this case

5 2 3 4 1 → 1 4 3 2 5 → 4 1 3 2 5

Flip 1 has to put 5 on bottom

Flip 2 must bring 4 to top.

$4 \le X \le 4$

Lower Bound

Upper Bound

X = 4

5th Pancake Number

P_5 = The number of flips required to sort the <u>worst case</u> stack of 5 pancakes.

Lower Bound

$? \le P_5 \le ?$

Upper Bound

5^{th} Pancake Number

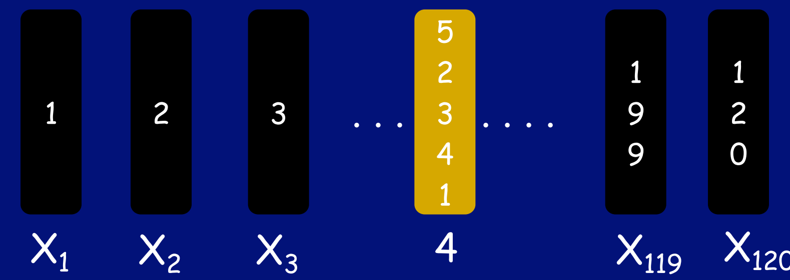
P_5 = The number of flips required to sort the worst case stack of 5 pancakes.

$4 \le P_5 \le ?$

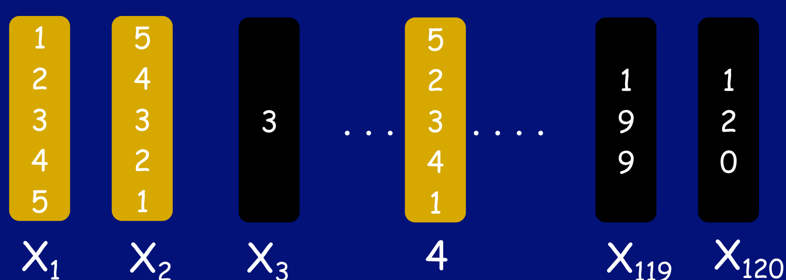
Lower Bound

Upper Bound

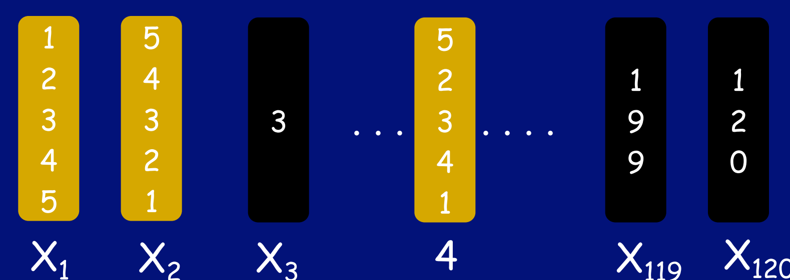
The 5^{th} Pancake Number: The MAX of the X's



The 5^{th} Pancake Number: The MAX of the X's



P_5 = MAX over s2 stacks of 5 of MIN # of flips to sort s



P_n =

MAX over s2 stacks of n pancakes
of MIN # of flips to sort s

P_n =

The number of flips required to sort the worst-case stack of n pancakes.

P_n =

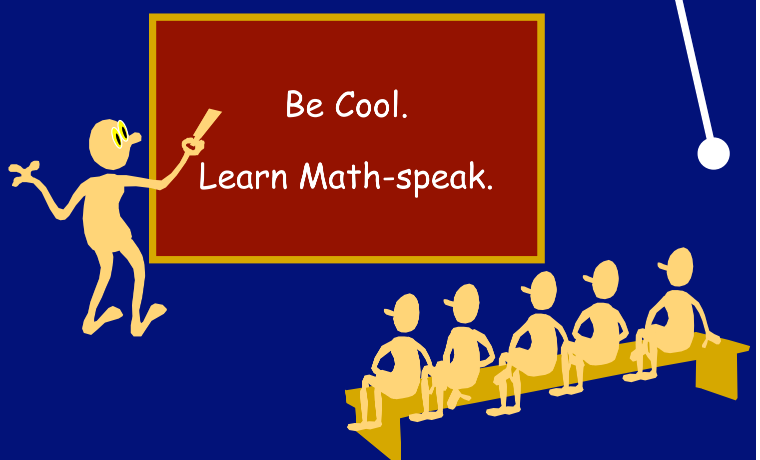
MAX over s2 stacks of n pancakes
of MIN # of flips to sort s

P_n =

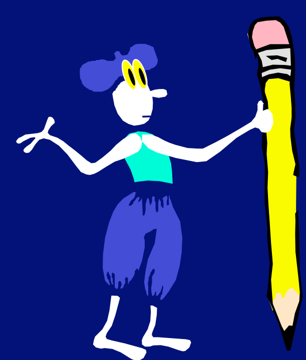
The number of flips required to sort a worst-case stack of n pancakes.

P_n =

The number of flips required to sort a worst-case stack of n pancakes.



What is P_n for small n?



Can you do

n = 0, 1, 2, 3 ?

Initial Values Of P_n

n	0	1	2	3
P_n	0	0	1	3

$P_3 = 3$

1
3
2

requires 3 Flips, hence $P_3 \geq 3$.

ANY stack of 3 can be done by getting the big one to the bottom ($\leq$ 2 flips), and then using $\leq$ 1 extra flip to handle the top two. Hence, $P_3 = 3$.

n^{th} Pancake Number

P_n = The number of flips required to sort a worst case stack of n pancakes.

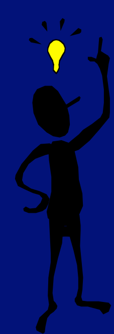
$$? \leq P_n \leq ?$$

Lower Bound

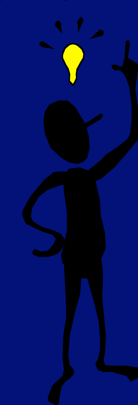
Upper Bound

Bracketing:

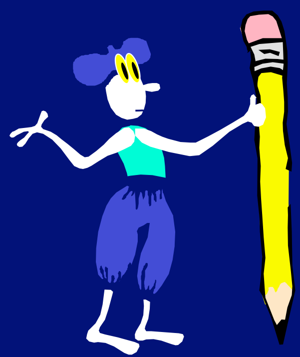
What are the best lower and upper bounds that I can prove?



$$\leq f(x) \leq$$



$? \le P_n \le ?$



Take a few minutes to try and prove upper and lower bounds on P_n, for $n > 3$.

Bring-to-top Method

Bring biggest to top.
Place it on bottom.

Bring next largest to top.
Place second from bottom.

And so on…

Upper Bound On P_n: Bring To Top Method For n Pancakes

If n=1, no work required - we are done!

Otherwise, flip pancake n to top and then flip it to position n.

Now use: Bring To Top Method For n-1 Pancakes

Total Cost: at most $2(n-1) = 2n - 2$ flips.

Better Upper Bound On P_n: Bring To Top Method For n Pancakes

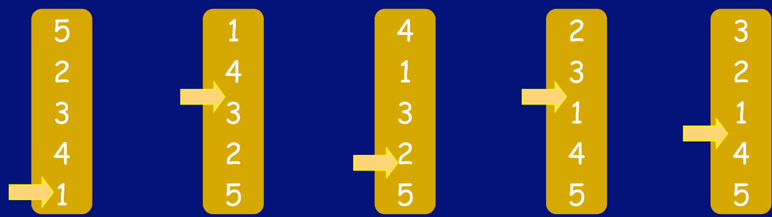
If n=2, at most one flip and we are done.

Otherwise, flip pancake n to top and then flip it to position n.

Now use: Bring To Top Method For n-1 Pancakes

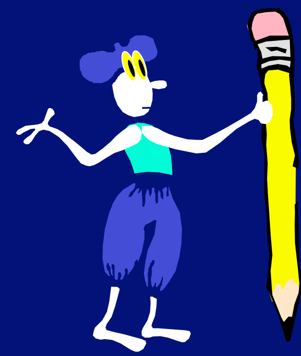
Total Cost: at most $2(n-2) + 1 = 2n - 3$ flips.

Bring to top not always optimal for a particular stack



Bring-to-top takes 5 flips, but we can do in 4 flips

$? \leq P_n \leq 2n-3$

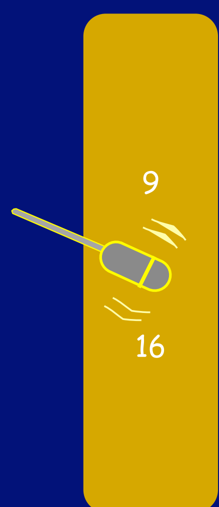


What other bounds can you prove on P_n?

Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

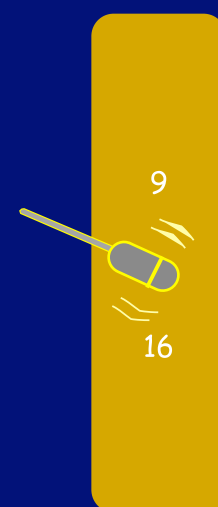


Breaking Apart Argument

Suppose a stack S contains a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack S must involve one flip that inserts the spatula between that pair and breaks them apart.

Furthermore, this same principle is true of the “pair” formed by the



$n \le P_n$

Suppose n is even.
S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

S

2
4
6
8
.
.
n
1
3
5
7
.
.
n-1

$n \le P_n$

Suppose n is even.
S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

Detail: This construction only works when $n>2$

S

2
1

$n \le P_n$

Suppose n is odd.
S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

S

1
3
5
7
.
.
n
2
4
6
8
.
.
n-1

$n \le P_n$

Suppose n is odd.
S contains n pairs that will need to be broken apart during any sequence that sorts stack S.

Detail: This construction only works when $n>3$

S

1
3
2

$n \leq P_n \leq 2n - 3$ for $n \geq 3$

Bring To Top is within a factor of two of optimal!

$n \leq P_n \leq 2n - 3$ for $n \geq 3$

Starting from ANY stack we can get to the sorted stack using no more than P_n flips.

From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$?

Reverse the sequences we use to sort.

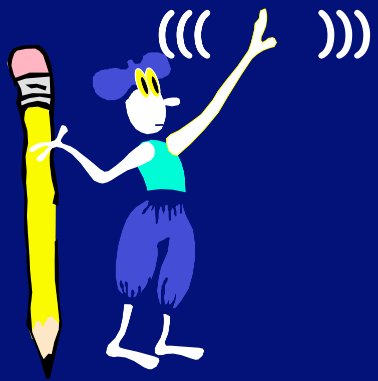
From ANY stack to sorted stack in $\leq P_n$.

From sorted stack to ANY stack in $\leq P_n$?

Hence,

From ANY stack to ANY stack in $\leq 2P_n$.

From ANY stack to ANY stack in ≤ $2P_n$.



Can you find a faster way than $2P_n$ flips to go from ANY to ANY?

From ANY Stack S to ANY stack T in ≤ P_n

Rename the pancakes in S to be 1,2,3,..,n.
Rewrite T using the new naming scheme that you used for S.
T will be some list: $\pi(1),\pi(2),..,\pi(n)$.

The sequence of flips that brings the sorted stack to $\pi(1),\pi(2),..,\pi(n)$ will bring S to T.

S:
4,3,5,1,2
1,2,3,4,5

T:
5,2,4,3,1
3,5,1,2,4

The Known Pancake Numbers

n	P_n
1	0
2	1
3	3
4	4
5	5 7
6	8
7	9
8	10
9	11
10	13
11	14
12	15
13	

P_{14} Is Unknown

14! Orderings of 14 pancakes.

14! = 87,178,291,200

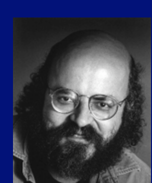
Is This Really Computer Science?

Posed in *Amer. Math. Monthly* 82 (1) (1975),
"Harry Dweighter" a.k.a. Jacob Goodman

$(17/16)n \leq P_n \leq (5n+5)/3$

William Gates &
Christos Papadimitriou

Bounds For Sorting By Prefix Reversal.
Discrete Mathematics,
vol 27, pp 47-57, 1979.



$(15/14)n \leq P_n \leq (5n+5)/3$

H. Heydari &
H. I. Sudborough

On the Diameter of the Pancake Network.
Journal of Algorithms,
vol 25, pp 67-94, 1997.

Permutation

Any particular ordering of all n elements of an n element set S, is called a permutation on the set S.

Example: S = {1, 2, 3, 4, 5}

One possible permutation: 5 3 2 4 1

5*4*3*2*1 = 120 possible permutations on S

Permutation

Any particular ordering of all n elements of an n element set S, is called a permutation on the set S.

Each different stack of n pancakes is one of the permutations on [1..n].

Representing A Permutation

We have many ways to specify a permutation on S. Here are two methods:

1) We list a sequence of all the elements of [1..n], each one written exactly once.
   Ex: 6 4 5 2 1 3

2) We give a function π on S such that $\pi(1)\ \pi(2)\ \pi(3)\ ..\ \pi(n)$ is a sequence that lists [1..n], each one exactly once.
   Ex: $\pi(1)=6$ $\pi(2)=4$ $\pi(3) = 5$ $\pi(4) = 2$ $\pi(4) = 1$ $\pi(6) =3$

A Permutation is a NOUN

An ordering S of a stack of pancakes is a permutation.

A Permutation is a NOUN.
A permutation can also be a VERB.

An ordering S of a stack of pancakes is a permutation.

We can permute S to obtain a new stack S'.

<u>Permute</u> also means to rearrange so as to obtain a permutation of the original.

Permute A Permutation.

I start with a permutation S of pancakes.
I continue to use a flip operation to permute my current permutation, so as to obtain the sorted permutation.

There are n! = 1*2*3*4*…*n permutations on n elements.

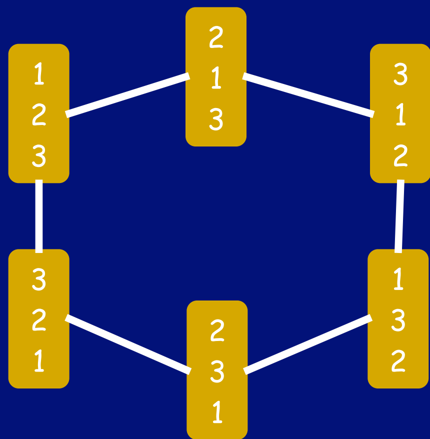
Easy proof in a later lecture.

Pancake Network: Definition For n! Nodes

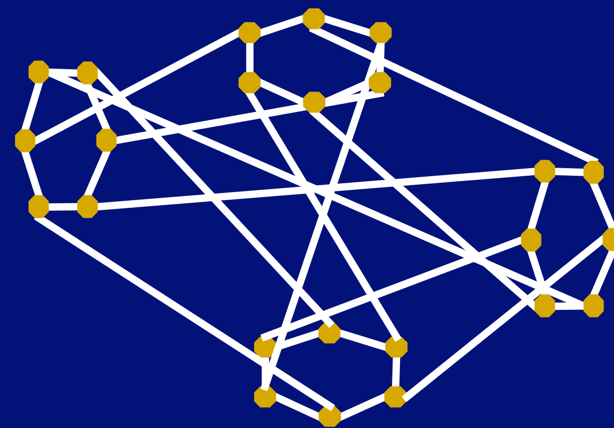
For each node, assign it the name of one of the n! stacks of n pancakes.

Put a wire between two nodes if they are one flip apart.

Network For n=3

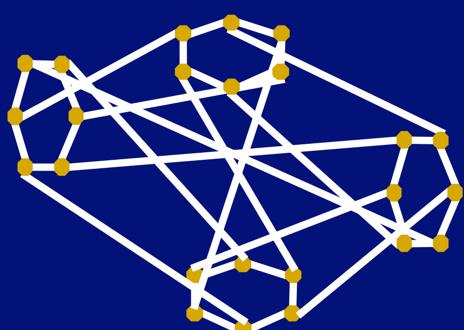


Network For n=4



Pancake Network: Message Routing Delay

What is the maximum distance between two nodes in the network?



P_n

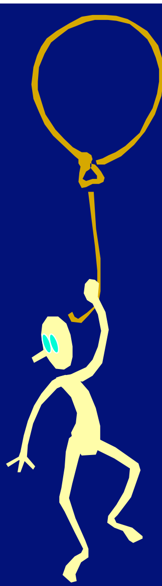
Pancake Network: Reliability

If up to n-2 nodes get hit by lightning the network remains connected, even though each node is connected to only n-1 other nodes.

The Pancake Network is optimally reliable for its number of edges and nodes.

Mutation Distance

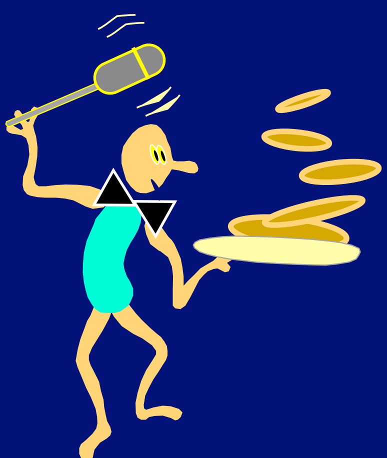
High Level Point



Computer Science is not merely about computers and programming, it is about mathematically modeling our world, and about finding better and better ways to solve problems.

This lecture is a microcosm of this exercise.

One "Simple" Problem



A host of problems and applications at the frontiers of science

Study Bee

Definitions of:

- nth pancake number
- lower bound
- upper bound
- permutation

Proof of:

- ANY to ANY in ≤ P_n

Important Technique:

- Bracketing

References

Bill Gates & Christos Papadimitriou: Bounds For Sorting By Prefix Reversal. *Discrete Mathematics,* vol 27, pp 47-57, 1979.

H. Heydari & H. I. Sudborough: On the Diameter of he Pancake Network. *Journal of Algorithms,* vol 25, pp 67-94, 1997