CS 19: Discrete Mathematics

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Today: Sets and Operations on Sets

The Language of Mathematics

- Math is not just a tool, it is a way of communicating ideas. It is a language.
- Before studying discrete math, we need to be comfortable with this language.
- The building blocks of this language:
  - Sets
  - Integers
    - More general numbers (rational, real, complex, …)
  - Functions
  - Logic
- More advanced math: sets are the only building blocks we need, all else can be built from sets.

Review of Sets

A set is a collection of distinct objects.

- These objects are called the elements or members of the set.
- If x is an element of the set S, we write $x \in S$ (read: "x belongs to S" or "x is in S").

Example: the set of fruits I like,

F = {apple, banana, fig, mango, pear}

- apple is an element of F, so apple $\in$ F.
- apricot is not, so apricot $\notin$ F.

Review of Sets, II

An element either belongs to a set or does not.

- No such thing as "partially belonging" to a set.
- No such thing as "number of copies" of an element in a set.

Example: set of initial letters of our first names.

Amit	Christopher	Clea	David
Emily	Evan	Gabriel	Grace
Jeff	Jeffrey	Jeremy	John
Jonathon	Matthew	Michael	Oleg
Ray	Taniquea	Tiger	Tom

{ A, C, D, E, G, J, M, O, R, T }

Finite and Infinite Sets

- The sets seen so far are all finite sets.
- We can count the number of elements in any one of these sets. This number is called the cardinality of the set.
- Cardinality of a set S is written as |S|.
- E.g., F = {apple, banana, fig, mango, pear} results in |F| = 5.
- Not all sets are finite. E.g., the set of all even natural numbers E = {2,4,6,8,10,12,...} is an infinite set. We could write |E| = ∞.

Other Interesting Possibilities

- A set can have just one element, e.g. {5}. Such a set is called a singleton set. Note that the set {5} is *not the same* as the integer 5.
- A set may have no elements at all! This very special set is called the empty set, and denoted ∅ or { }.
- The elements of a set might themselves be sets, e.g. { {A,H,I,M,O,T,U,V,W,X,Y}, {B,C,D,E,H,I,K,O,X}, {H,I,O,X} }.
- Quick quiz: what is the cardinality of the above set?
- Answer: 3

Describing a Set

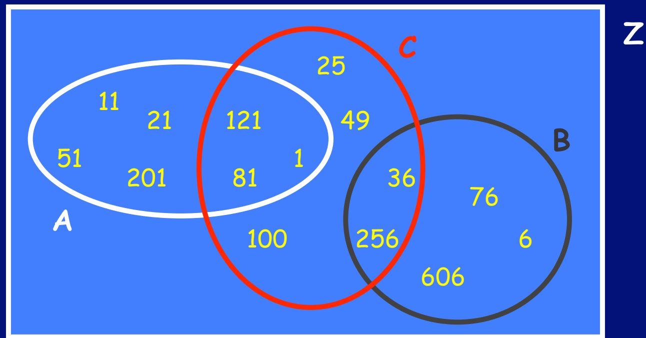
There are two main ways:

1. Roster notation: list out all the elements
   - F = {apple, banana, fig, mango, pear}
2. Set-builder notation: describe the property of a generic element of the set and write {x : some property of x}.
   - F = { x : x is a fruit and I like x }
   - Read: "F is the set of all x such that x is a fruit and I like x."

Describing Large or Infinite Sets

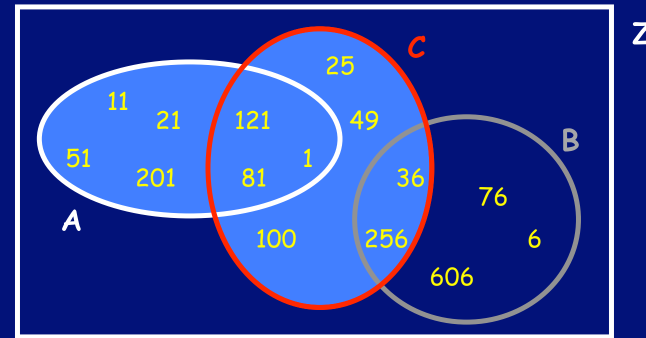
- For informal descriptions, using an ellipsis is fine, e.g. {2,4,6,8,...} or {0,1,2,...,99}.
- But in formal mathematical writing (such as in this course), you should use set-builder notation.
- This avoids ambiguity, e.g., when you see {1,3,5,7,...} is it
  - {1,3,5,7,9,11,13,...} = {x : x is an odd positive integer} ?
  - {1,3,5,7,11,13,17,...} = {x : x is an odd prime number} ?
  - {1,3,5,7,11,13,15,17,21,23,25,27,...} = {y : y is an odd number that does not end in 9} ?

Depicting Sets: Venn Diagrams



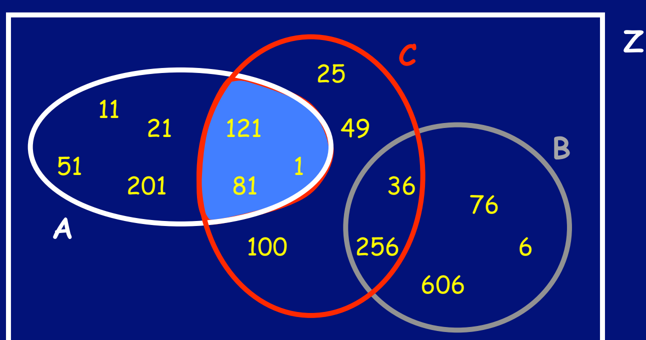
$A = \{1,11,21,51,81,121,201\}$, $B = \{6,36,76,256,606\}$
$C = \{1,25,36,49,81,100,121,256\}$, Z = set of all integers

Operations on Sets: Union



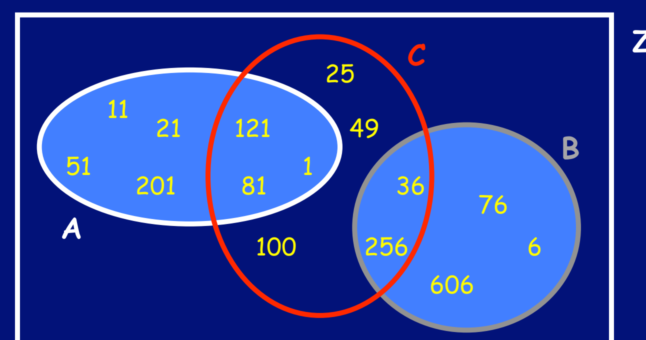
$A \cup C = \{1, 11, 21, 25, 36, 49, 51, 81, 100, 121, 201, 256\}$
$S \cup T = \{x : \text{either } x \in S \text{ or } x \in T\}$

Operations on Sets: Intersection



$A \cap C = \{1, 81, 121\}$
$S \cap T = \{x : x \in S \text{ and } x \in T\}$

Disjoint Sets



Two sets with no elements in common, e.g. A & B above.
Mathematically, S and T are disjoint if $S \cap T = \varnothing$.

Subsets

- If a set A lies completely "within" a set B, then we say that A is a subset of B. We write $A \subseteq B$.
- Note: some authors write $A \subset B$, but I recommend that you avoid this.
- Mathematically, $A \subseteq B$ means that every element of A is also an element of B. In other words, if $x \in A$ then $x \in B$.

- In Venn diagram terms, the blob representing A must sit inside the blob representing B.
- Quick quiz: are there any subset relationships between A, $A \cup B$ and $A \cap B$?

Basic Facts about Sets

For any two sets A and B,

- $A \subseteq A \cup B$

  Proof: Suppose $x \in A$.
  Then, either $x \in A$ or $x \in B$.
  Therefore, $x \in A \cup B$.
- $A \cap B \subseteq A$

  Proof: Suppose $x \in A \cap B$.
  Then, $x \in A$ and $x \in B$.
  Therefore, $x \in A$.
- $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

  Proof: Left as an exercise.

Basic Facts about Sets, II

- For any set A, we have $A \subseteq A$.
- If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - In fact, this is the only rigorous way to prove that two sets A and B are equal. Break your proof into two parts. First prove $A \subseteq B$. Then prove $B \subseteq A$.
- For any set A, we have $\varnothing \subseteq A$.
  - Why does this make logical sense?
  - Definition says "if $x \in \varnothing$ then $x \in A$", but $\varnothing$ is the empty set so we can never have $x \in \varnothing$.
  - We'll revisit this issue when we study logic. For now, think in terms of Venn diagrams.

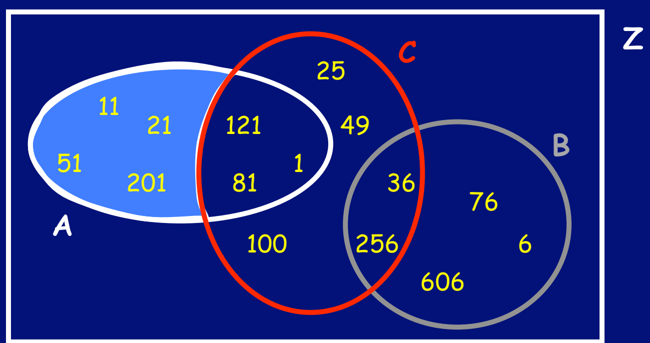
Quick Quiz

What are the subsets of {1,2,3} ?

- { },
- {1}, {2}, {3},
- {1,2}, {2,3}, {1,3},
- {1,2,3}

- How many subsets in total? … 8
- How many subsets does {apple,fig,mango} have?

Operations on Sets: Difference



$A - C = \{11, 21, 51, 201\}$

$S - T = \{x : x \in S \text{ and } x \notin T\}$

More Facts about Sets

For any two sets A and B,

- $A - B \subseteq A$.
- The sets B and $(A - B)$ are disjoint.
- If $A \cap B = \varnothing$, then $A - B = A$.

Many interesting distributive laws, e.g.

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $(A - B) \cap C = (A \cap C) - (B \cap C)$.
- But beware!! $(A - B) \cup C \neq (A \cup C) - (B \cup C)$.

Cartesian Product

A more sophisticated operation on sets: produces a set of "ordered pairs" (a,b).

$$A \times B = \{ (a,b) : a \in A \text{ and } b \in B \}.$$

black	(table, black)	(pen, black)	(hair, black)
red	(table, red)	(pen, red)	(hair, red)
	table	pen	hair

e.g., if A = {table, pen, hair}, B = {red, black}, then
A × B = { (table, red), (table, black), (pen, red), (pen, black), (hair, black), (hair, black) }.

Set versus Ordered Pair

Important! A set is inherently *unordered*. Therefore, $\{a, b\} = \{b, a\}$.

However, for an ordered pair, the *order matters*. Therefore, $(a, b) \neq (b, a)$.

- Unless, of course, $a = b$.

Also, note that (x, x) is a perfectly good ordered pair, whereas $\{x, x\}$ is confusing, because a set cannot contain "two copies" of x.

What is the Cartesian Product Good For?

For constructing sets whose elements can be "decomposed" into simpler parts.

E.g., an integer can be thought of as having two parts: a sign and a magnitude. Thus, the set of integers can be thought of as

$$\{+,-\} \times \{0,1,2,3,\ldots\}$$
$$= \{ (+,0), (-,0), (+,1), (-,1), (+,2), (-,2), \ldots \}$$

Slight annoyance: (+,0) and (-,0) need to be "merged."

What is the Cartesian Product Good For?

For constructing sets whose elements can be "decomposed" into simpler parts.

E.g., a rational number can be thought of as having two parts: a numerator and a denominator. Thus, if Z denotes the set of integers, then the set of rational numbers can be thought of as $Z \times (Z - \{0\})$.

ordered pair (4,7) → rational number 4/7.

Not-so-slight annoyance: lots of pairs need to be "merged", e.g., (1,3), (2,6), (8,24), (100,300), ……

More Fun with Cartesian Product

Cartesian product gives us a tool to connect two (or more) sets, via relations and functions.

But that's another story, for another class.

Study Bee

Concepts:

- Sets, finite and infinite
- Roster and set-builder notations
- Venn diagrams
- Union, intersection, disjoint sets
- Subsets
- Difference
- Cartesian product

Theorems:

- Some basic properties of set operations and subsets
- Distributive laws