CS 19: Discrete Mathematics

Amit Chakrabarti

Logic: Variables & Quantifiers

Summary of Last Lecture

- Propositions (= declarative sentences, true/false)
- Logical connectives

  $\wedge$: "and" $\vee$: "or" $\neg$: "not"

  $\oplus$: "xor" $\Rightarrow$: "implies" $\Leftrightarrow$: "iff"
- Truth tables
- Verifying logical equivalence via truth tables

Predicate

A predicate is a declarative statement involving one or more variables (i.e., unknown quantities).

- "x is a perfect square"
  - » where x represents an integer
- "$3x - y > 5$"
  - » where x and y represent real numbers
- "s has height less than 6 feet"
  - » where s represents a student in this class

To determine truth, need value of variable(s)

Predicate: Definition

Let S be any nonempty set.

A predicate on S is a function

$$P : S \rightarrow \{\text{true}, \text{false}\} .$$

In our earlier examples

- "x is a perfect square" was a predicate on $\mathbf{Z}$.
- "s has height less than 6 feet" was a predicate on the set of students in this class.
- "$3x - y > 5$" was a predicate on … ???

Functions with Multiple Inputs

If f is a function that takes two integers as input and produces an integer output, then use

$$f : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$$

Thus, f(x,y) really means f((x,y)).

- Using this trick for predicates, we can think of "3x - y > 5" as a predicate on **R** × **R**.

Predicates to Propositions

Suppose P is a predicate on the set S.

The obvious way to make a proposition out of P is to plug in a value from S.

- "x is a perfect square"
  - » Plug in x = 5. Get "5 is a perfect square".
- "3x - y > 5"
  - » Here, the variable is the ordered pair (x,y).
  - » Plug in (x,y) = (2.4, 0.76). Get "3 × 2.4 – 0.76 > 5".

But there is another interesting way…

Predicates to Propositions, II

Suppose P is a predicate on the set S.

Instead of plugging a value into P, we could ask:

- Is P always true?
  - "x is a perfect square" … is that always true, for all integers x?
- Is P ever true?
  - "x is a perfect square" … is that ever true, for any integer x?

Quantifiers

Such propositions come up all the time in mathematics (and computer science), so we have special notation for them.

$\forall$: "for all"

$\exists$: "there exists"

These symbols are called quantifiers.

These are the two most important symbols to learn in this course!

Using Quantifiers

Let P be a predicate on **Z**, given by P(x) = "x is a perfect square".

- "P is always true" is written as "$\forall x\, P(x)$".
  - Pronounced "for all x, P(x)".
  - $\forall$ is called the universal quantifier.
- "P is sometimes true" is written as "$\exists x\, P(x)$".
  - Pronounced "there exists x such that P(x)".
  - $\exists$ is called the existential quantifier.

The Universe Matters!

The notation "$\forall x\, P(x)$" does not make it clear what the domain of x (a.k.a. the universe) is. But this is crucial info:

- $\forall x\, (2x \geq x)$
  - True if the universe is the set of natural numbers
  - False if the universe is **Z**.
- $\forall x\, \exists y\, (y = x/5)$
  - True if universe is **R**, false if universe is **Z**.

Specifying a Universe

Two possible ways to disambiguate

1. Specify the universe at the outset. E.g., "in this solution, quantifiers are over the universe of all natural numbers."
2. Specify the universe when using the quantifier(s). E.g.,
   - $\forall x \in \mathbf{N}\, (2x \geq x)$
   - $\forall x \in \mathbf{Z}\, (\exists y \in \mathbf{R}\, (y = x/5))$

Negation

A compound proposition can be systematically negated by using two simple rules:

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

These are called De Morgan's Laws.

Exercise: verify these using truth tables.

Negating Quantified Statements

Basically an extension of De Morgan's Laws. Think of

- $\forall x \in S\ (P(x))$ as a big AND over all elements of S.
- $\exists x \in S\ (P(x))$ as a big OR over all elements of S.

Then, we see that

- $\neg\ \forall x \in S\ (P(x)) \Leftrightarrow \exists x \in S\ (\neg\ P(x))$
- $\neg\ \exists x \in S\ (P(x)) \Leftrightarrow \forall x \in S\ (\neg\ P(x))$

Very important to internalize these rules!