Homework 4
Due Tue Dec 6, 5:00pm

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General Instructions. Each problem has a fairly short solution. Feel free to reference things we have proved in class, to keep your own solutions short. **Each problem is worth 10 points.**

Honor Prinicple. For this homework, you may only refer to lecture notes from this class. You are allowed to discuss the problems and exchange solution ideas with your classmates. But when you write up any solutions for submission, you must work alone. *If in doubt, ask the professor for clarification!*

17. Prove that, for every Boolean function $f : \{0,1\}^n \to \{0,1\}$, we have $\text{depth}(f) = \Theta(\log \mathrm{L}_D(f))$, where L_D denotes de Morgan formula length.

    Hint: One direction should be very easy, using induction (say). For the other direction, first prove that, in a rooted binary tree with ℓ leaves, there exists a node v such that the number of leaves that are descendants of v lies in $[\ell/3, 2\ell/3]$. Then use this fact to "balance" a formula that is needlessly deep.

18. Recall the Karchmer-Wigderson game R_f corresponding to a nonconstant Boolean function $f : \{0,1\}^n \to \{0,1\}$. Alice gets an input $x \in \{0,1\}^n$ such that $f(x) = 1$ and Bob gets an input $y \in \{0,1\}^n$ such that $f(y) = 0$. Alice and Bob must agree on an index $i \in [n]$ such that $x_i \neq y_i$. Prove that $\mathrm{D}(R_f) = \text{depth}(f)$, where 'D' denotes deterministic communication complexity and 'depth' denotes circuit depth.

    Hint: We have already shown in class that $\mathrm{D}(R_f) \le \text{depth}(f)$. For the other direction, use induction on the complexity of the best protocol for the game.

19. Let $\textsc{ustcon}_N : \{0,1\}^{\binom{N}{2}} \to \{0,1\}$ be the "undirected s–t connectivity" Boolean function. That is, the input is viewed as the description of an *undirected* labeled N-vertex graph, and the output is 1 iff there is a path in this graph between two designated vertices s and t (WLOG, we may take $s = 1, t = 2$).

    Prove the monotone circuit depth lower bound $\text{depth}_m(\textsc{ustcon}_N) = \Omega(\log^2 N)$. You may follow the same outline as our work in class for the function $\textsc{stcon}$, so you don't have to repeat long proofs from class. Just highlight the differences. Be explicit about what you need to change in the proof.

    Hint: Consider the relation $\textsc{fork}' \in [w]^\ell \times [w]^\ell \times \{0,1,\ldots,\ell\}$ given by

    $$(x, y, i) \in \textsc{fork}' \iff \tilde{x}_i = \tilde{y}_i \wedge (\tilde{x}_{i-1} \neq \tilde{y}_{i-1} \vee \tilde{x}_{i+1} \neq \tilde{y}_{i+1}),$$

    where $\tilde{x} = 1 \circ x \circ 1$ and $\tilde{y} = 1 \circ y \circ 2$.

20. Recall the function $\textsc{ed}_n$ from the previous homework. Using Neciporuk's method, suitably adapted for branching program size, show that $\mathrm{BP}(\textsc{ed}_n) = \Omega(n^2/\log^2 n)$.