# Solutions: Homework 3

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## 1

1.16(a) <u>Answer</u>:

$R^0_{11} = \varepsilon \cup a,\ R^0_{22} = \varepsilon \cup a,\ R^0_{12} = b,\ R^0_{11} = b.$

$R^1_{12} = R^0_{12} \cup R^0_{11}(R^0_{11})^* R^0_{12} = a^* b$

$R^1_{22} = R^0_{22} \cup R^0_{21}(R^0_{11})^* R^0_{12} = (\varepsilon \cup a) \cup ba^* b$

$L = R^2_{12} = R^1_{12} \cup R^1_{12}(R^1_{22})^* R^1_{22} = a^* b((a \cup \varepsilon)ba^* b)^*$

1.16(b) $R^0_{11} = \varepsilon,\ R^0_{22} = a \cup \varepsilon,\ R^0_{33} = \epsilon,\ R^0_{12} = (a \cup b),\ R^0_{23} = b,\ R^0_{13} = \phi,\ R^0_{21} = \phi,\ R^0_{32} = b,$
$R^0_{31} = a.$

$R^1_{11} = R^0_{11} \cup R^0_{11}(R^0_{11})^* R^0_{11} = \varepsilon,$

$R^1_{22} = R^0_{22} \cup R^0_{21}(R^0_{11})^* R^0_{12} = a \cup \varepsilon,$

$R^1_{33} = R^0_{33} \cup R^0_{31}(R^0_{11})^* R^0_{13} = \varepsilon,$

$R^1_{12} = R^0_{12} \cup R^0_{11}(R^0_{11})^* R^0_{12} = (a \cup b),$

$R^1_{23} = R^0_{23} \cup R^0_{21}(R^0_{11})^* R^0_{13} = b,$

$R^1_{13} = R^0_{13} \cup R^0_{11}(R^0_{11})^* R^0_{13} = \phi,$

$R^1_{21} = R^0_{21} \cup R^0_{21}(R^0_{11})^* R^0_{11} = \phi,$

$R^1_{32} = R^0_{32} \cup R^0_{31}(R^0_{11})^* R^0_{12} = b \cup a(a \cup b),$

$R^1_{31} = R^0_{31} \cup R^0_{31}(R^0_{11})^* R^0_{11} = a.$

$R^2_{11} = R^1_{11} \cup R^1_{12}(R^1_{22})^* R^1_{21} = \varepsilon,$

$R^2_{13} = R^1_{13} \cup R^1_{12}(R^1_{22})^* R^1_{23} = (a \cup b)a^* b,$

$R^2_{33} = R^1_{33} \cup R^1_{32}(R^1_{22})^* R^1_{23} = \varepsilon \cup (b \cup a(a \cup b))a^* b,$

$R^2_{12} = R^1_{12} \cup R^1_{12}(R^1_{22})^* R^1_{22} = (a \cup b)a^*,$

$R^2_{21} = R^1_{21} \cup R^1_{22}(R^1_{22})^* R^1_{21} = \phi,$

$R^2_{31} = R^1_{31} \cup R^1_{32}(R^1_{22})^* R^1_{21} = a,$

$R^2_{22} = R^1_{22} \cup R^1_{22}(R^1_{22})^* R^1_{21} = a^*,$

$R^2_{32} = R^1_{32} \cup R^1_{32}(R^1_{22})^* R^1_{22} = (b \cup a(a \cup b))a^*,$

$R^3_{11} = \varepsilon \cup (a \cup b)a^* b((b \cup a(a \cup b))a^* b)^* a.$

$R^3_{13} = (a \cup b)a^* b((b \cup a(a \cup b))a^* b)^*.$

$L = R^3_{11} \cup R^3_{13}.$

## 2

2.1 Answer: False. For a counterexample, let $L$ be any non-regular language such as $\{0^n1^n \mid n > 0\}$. Since $L$ is non-regular, $\overline{L}$(the complement of $L$) is also nonregular. Yet, $L \cup \overline{L} = \Sigma^*$ is regular.

2.2 Answer: False. For a counterexample, let $L$ be any non-regular language such as $\{0^n1^n \mid n > 0\}$. Since $L$ is non-regular, $\overline{L}$ (the complement of $L$) is also nonregular. Yet, $L \cap \overline{L} = \Phi$ is regular.

2.3 Answer: True. We proved in class that if a language is regular, so is its complement. This implies the above statement.

2.4 Answer: False. For a counterexample, let $A_k = \{0^k1^k\}$. Clearly, for all $k > 0$, since $A_k$ is a finite language, it is regular. However, $A_1 \cup A_2 \cup A_3 \cup ...$ is $\{0^k1^k \mid k > 0\}$, which is nonregular.

2.5 (Note: A string $w$ is in the intersection if and only if it is in *every* $A_i$.)

Answer: False. For a counterexample, let $S = \{0^p \mid p \text{ is a prime }\}$. Define $A_k$ as $0^* - \{0^x \mid x \text{ is the } k\text{th nonprime }\}$. Clearly, each $A_k$ is regular, yet $A_1 \cap A_2 \cap A_3 \cap ...$ is $S$, which is nonregular.

## 3

Answer: Here is the intuition for how to construct a machine $M'$ for MAX($L$): If $q_f$ is a final state of $M$ and there is a non-empty string that drives $M$ from $q_f$ to a final state (possibly $q_f$ itself), then $q_f$ should not be a final state in $M'$. This ensures that $M'$ does not accept a string in $L$ if there is a way of extending it to be another string in $L$.

Formally, let DFA $M' = (Q',\ \Sigma,\ \delta',\ q_0',\ F')$, where:
$Q' = Q$
$q_0' = q_0$
$\delta' = \delta$
$F' = \{q \mid q \in F \text{ and for all non-empty strings } y,\ \hat{\delta}(q, y) \notin F\}$
Clearly $M'$ recognizes MAX($L$).

## 4

Answer: We observe that a string $w$ is in CYCLE($L$) if and only if there is a way of dividing $w$ into two parts $x_1$ and $x_2$ and there is a state $q$ of the original machine such that a marble starting off in state $q$ ends up in a final state of $M$ upon consuming $x_1$ and a marble starting off in the initial state of $M$ ends up in $q$ upon consuming $x_2$. This suggests that the marble should keep track of three things: (i) the state of $M$ where it started, (ii) which state of $M$ it currently is, and (iii) whether it is still consuming the first part($x_1$) of the input or the second part ($x_2$). Accordingly, each state of $M'$ will be of the form $[p, q, i]$ where $p$, $q$ are elements of $Q$ and i is one of 1 or 2.

Formally, let NFA $M' = (Q',\ \Sigma,\ \delta',\ q_0',\ F')$, where:
$Q' = \{q_0'\} \cup \{[p, q, i] \mid p, q \in Q,\ i \in \{1, 2\}\}$
$F' = \{[q, q, 2] \mid\ q \in Q\}$
$\delta'(q_0', \varepsilon) = \{[p, p, 1] \mid\ p \in Q\}$
For all $p,\ q \in Q,\ i \in \{1, 2\},\ a \in \Sigma$,
$\delta'([p, q, i],\ a) = \{[p,\ \delta(q, a),\ i]\}$
For all $q \in F,\ p \in Q$,
$\delta'([p, q, 1],\ \varepsilon) = \{[p, q_0, 2]\}$

# 5

5.1 $L = \{0^m 1^n 0^{m+n} \mid m,\ n$ are any natural numbers $\}$

Answer: $L$ is not regular. For a proof, assume $L$ is regular. Let $s = 0^p 1^p 0^{2p}$, where $p$ is the constant mentioned in PL. Clearly $s$ is in $L$ and $|s| \geq p$. So, by PL, there are $x,\ y,\ z$ such that $s = xyz,\ |xy| \leq p, |y| \geq 1, xy^i z$ is in $L$ for all $i \geq 0$. Since $|xy| \leq p$, $y$ lies entirely in the first sequence of 0's. Hence, $xy^2z = 0^{p+|y|}1^p 0^{2p}$. Since $|y| \geq 1$, it follows that $xy^2z$ is not in $L$, contradicting PL. We conclude that $L$ is regular.

5.2 $L = \{0^m 1^n \mid m$ divides $n\}$

Answer: $L$ is not regular. For a proof, assume $L$ is regular. Let $p$ be the constant mentioned in PL. Let $s = 0^q 1^q$, where $q$ is a prime number greater than $p + 1$. Clearly $s$ is in $L$ and $|s| \geq p$. So, by PL, there are $x,\ y,\ z$ such that $s = xyz,\ |xy| \leq p,\ |y| \geq 1,\ xy^i z$ is in $L$ for all $i \geq 0$. Since $|xy| \leq p$, $y$ lies entirely in the first sequence of 0's. Hence, $xy^0z = xz = 0^{q-|y|}1^q$. Since $p \geq |y| \geq 1$ and $q \geq p + 2$, it follows that $q - 1 \geq q - |y| \geq 2$. This, together with the fact that $q$ is a prime, implies that $q - |y|$ does not divide $q$. Thus, $xz$ is not in $L$, contradicting PL. We conclude that $L$ is not regular.

5.3 $\{xwx^R \mid\ x$ and $w$ are strings in $(0 \cup 1)^+\}$

Answer: This language is regular. To see this, note that a string is in $L$ if and only if it begins and ends with the same symbol, and has at least three symbols. Thus, the following regular expression captures $L$: $0(0 \cup 1)^+0\ \cup\ 1(0 \cup 1)^+1$. Since $L$ has a regular expression, $L$ is regular.

5.4 $\{0^m \mid m = 2^n$ for some natural number $n\}$

Answer: $L$ is not regular. For a proof, assume $L$ is regular. Let $s = 0^{2^p}$, where $p$ is the constant mentioned in PL. Clearly $s$ is in $L$ and $|s| \geq p$. So, by Pumping Lemma, there are $x, y, z$ such that $s = xyz,\ |xy| \leq p,\ |y| \geq 1,\ xy^i z$ is in $L$ for all $i \geq 0$. We have:

$2^p\ < |xy^2z|$ (because $|xyz| = 2^p$ and $|y| \geq 1$)

$\leq 2^p + p$  (because $|xyz| = 2^p$ and $|xy| \leq p$)

$< 2^p + 2^p$  (because $p < 2^p$ for any $p \geq 1$)

$= 2^{p+1}$

It follows that $|xy^2z|$ is not a power of 2, and so $xy^2z$ is not in $L$. This contradicts PL. We conclude that $L$ is not regular.

5.5 problem 1.28 in the book.

Answer: The language $E$ is not regular. For a proof, assume $E$ is regular. Let $p$ be the constant mentioned in PL. Consider string $s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^p \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}^p$. Clearly $s \in E$ and $|s| \geq p$, therefore, by PL, there exist $x, y, z$ such that $s = xyz$, $|xy| \leq p$, $|y| \geq 1$, and $xy^iz \in E$ for all $i \geq 0$. Since $|xy| \leq p$, it follows that $y$ lies entirely in the first sequence of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}^i$. Therefore, we have $xy^2z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{p+|y|} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}^p$. Since $|y| \geq 1$, it follows that $xy^2z \notin E$, contradicting PL. Hence, $E$ is not regular.

5.6 $\{0^m1^n \mid m \text{ is not equal to } n\}$

Answer: $L$ is not regular. For a proof, let $A$ denote the language $\{0^n1^n \mid n \geq 0\}$. We observe that $A = \overline{R \cup L}$, where $R$ is the complement of $0^*1^*$. Since $0^*1^*$ is regular, its complement, namely, $R$ is regular. If $L$ were regular, then since regular languages are closed under union, $R \cup L$ would be regular. Since regular languages are closed under complementation, it follows that $\overline{R \cup L}$, which is $A$, is regular, contradicting the well known fact that $A$ is not regular. We conclude that $L$ is not regular.