

Please think carefully about how you are going to organise your answers *before* you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Do both parts of Exercise 1.16 from the textbook. Use the  $R_{ij}^k$  method as described in the lecture notes for the Oct 11 lecture; do not use the textbook's "GNFA method." Try to simplify the intermediate regular expressions, so as to make your own life simple! [6+7 points]
  
2. Are the following statements *always* true? If true, give a brief justification and if false, give a concrete counterexample. Below,  $A$  and  $B$  denote languages over some alphabet  $\Sigma$ .
  - 2.1. If  $A \cup B$  is regular, then at least one of  $A$  and  $B$  is regular. [5 points]
  
  - 2.2. If  $A \cap B$  is regular, then at least one of  $A$  and  $B$  is regular. [5 points]
  
  - 2.3. If  $\overline{A}$  (defined as  $\Sigma^* - A$ ) is regular, then  $A$  is regular. [5 points]
  
  - 2.4. A union of arbitrarily many regular languages is regular, even if it is an infinite union. [5 points]
  
  - 2.5. An intersection of arbitrarily many regular languages is regular, even if it is an infinite intersection. [5 points]
  
3. For a language  $L$  over alphabet  $\Sigma$ , define  $\text{MAX}(L) = \{x \in \Sigma^* : x \in L \text{ and } x \text{ is not a proper prefix of any string in } L\}$ . Prove that if  $L$  is regular, then so is  $\text{MAX}(L)$ . [8 points]
  
4. For a language  $L$  over alphabet  $\Sigma$ , define  $\text{CYCLE}(L) = \{x_1x_2 : \exists x_1, x_2 \in \Sigma^* \text{ such that } x_2x_1 \in L\}$ . Prove that if  $L$  is regular, then so is  $\text{CYCLE}(L)$ . [12 points]

Continued overleaf..

5. For each of the following languages, say whether or not the language is regular and prove your answer. To prove that a language is regular, specify a finite automaton or a regular expression for that language. To prove that a language is not regular, use the pumping lemma or closure properties of regular languages.

Proofs *must* be precisely written. *Make sure you fully understand the definitions of the sets before answering.*

5.1.  $\{0^m 1^n 0^{m+n} : m, n \geq 0\}$ . [7 points]

5.2.  $\{0^m 1^n : m \text{ divides } n\}$ . [7 points]

5.3.  $\{xwx^R : x, w \in \{0, 1\}^*, |x| > 0 \text{ and } |w| > 0\}$ . [7 points]

5.4.  $\{0^{2^n} : n \geq 0\}$ . [7 points]

5.5. Problem 1.28 from the textbook. [7 points]

5.6.  $\{0^m 1^n : m, n \geq 0 \text{ and } m \neq n\}$ . [7 points]

### Challenge Problems

**CP3:** Let  $L$  be any subset of  $0^*$ . Prove that  $L^*$  is regular.

This is a delightful problem and will teach you something nice about regular languages if you solve it.