Solutions to Mid-Term Exam
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Discussion

It is always a pleasure to see students come up with elegant solutions and insights that had not occurred to me when I created the exam. I experienced this pleasure several times while grading your work. Here are the most notable examples.

- Nicolas Baum gave a very elegant solution to #4 using only algebra on sets. My official solution, as well as almost all correct solutions from you, argue the equality using the structure of strings in the two sets. This results in some notation soup. Nicolas's solution avoids this very beautifully.
- Nicolas Baum also came up with the most succinct and elegant PDA construction for #7.
- Naomi Forman gave a thorough and rigorous proof that her construction for #3 works, using the formal definition of acceptance of a string by a DFA/NFA.
- John Paul Lewicke solved #6.4 by making the brilliant observation $\overline{\text{SELECT}(\overline{L})} = \text{PERMUTE}(L)$. So if regular languages were closed under the SELECT operation, then by closure under complement they would also be closed under the PERMUTE operation; but this was shown to be false in #6.3. Another student also hit upon this idea, but had imperfect execution.
- Jason Reeves solved #8 using an unexpected (to me) approach and arrived at a short 3-variable CFG that I have reproduced without proof following my official solution. It takes a little work to see that it is correct, but it is definitely a cute solution.

Unfortunately, all is not rosy. The two most common sources of lost points were (1) failing to provide even a sketch of a proof for the correctness of your construction in #3, and (2) believing that #6.4 was a true statement. Also, a number of you lost a little bit on #6.3 for using $\text{PERMUTE}((01)^*) = \{x \in \{0,1\}^* : N_0(x) = N_1(x)\}$ but proving only one of the two inclusions.

For the most part, your proofs were at the right level of rigor. In the few cases where this was not so, I have comment specifically that your solution is not rigorous enough. **Please read these official solutions carefully** to absorb the level of rigor expected in this course. Please feel free to have a chat with me or your TA if you remain confused about this rigor issue.

Overall I would rate the class's work on the midterm as "very good." With a little more care your work on the final would be "excellent."

Please note that Khanh Do Ba graded #1, #2, #3, #7 and #8 and that I graded the rest.

The Solutions

1. Write a regular expression for the language generated by the following grammar:

$$\begin{aligned} S &\longrightarrow AT \\ T &\longrightarrow ABT \mid TBA \mid AA \\ A &\longrightarrow 0 \\ B &\longrightarrow 1 \end{aligned}$$

A single line answer will do; you don't have to justify or show any steps. Your regular expression should be as simple as possible.

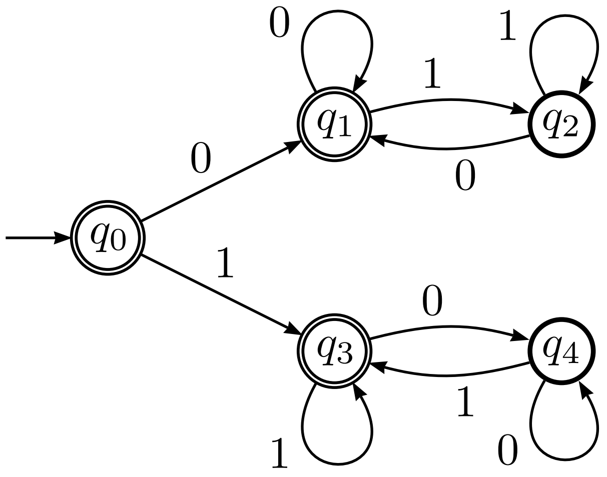
Solution: The grammar generates $0(01)^*00(10)^*$.

2. Draw a DFA for the language

$$\{x \in \{0,1\}^* : x \text{ contains an equal number of occurrences of the substrings } 01 \text{ and } 10\}\,.$$

For example, 101 and 0000 are in the language, but 1010 is not.

Solution: The idea is to handle strings beginning with a 1 and strings beginning with a 0 separately. The following DFA does the job:



3. Recall that $x^{\mathcal{R}}$ denotes the reverse of the string x. For a language L, let $L^{\mathcal{R}} = \{x^{\mathcal{R}} : x \in L\}$. Prove that if L is regular, so is $L^{\mathcal{R}}$.

Solution: The idea is to start with a DFA for the regular language L and "reverse all the arrows" in this DFA, make its start state an accept state of the resulting machine, and make all its accept states start states (or rather, since only one start state is allowed, to introduce ε-transitions from a new start state to all the former accept states). This procedure clearly gives us an NFA, because, for instance, if there are five different states of the initial DFA that all lead into a state q on reading input symbol a, then after the conversion, q will lead to these five different states on input a. Now we give a formal proof.

Suppose L is a regular language. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing L. Let us construct an NFA $M' = (Q \cup \{q_{\text{new}}\}, \Sigma, \delta', q_{\text{new}}, \{q_0\})$ where δ' is given by

$$\begin{aligned}
\delta'(q_{\text{new}}, \varepsilon) &= F\,, & \\
\delta'(q_{\text{new}}, a) &= \emptyset\,, & \forall\, a \in \Sigma\,, \\
\delta'(q, \varepsilon) &= \emptyset\,, & \forall\, q \in Q\,, \\
\delta'(q, a) &= \{r \in Q : \delta(r, a) = q\}\,, & \forall\, q \in Q,\ a \in \Sigma\,.
\end{aligned}$$

We claim that $\mathcal{L}(M') = L^{\mathcal{R}}$, which would prove that $L^{\mathcal{R}}$ is regular. To prove our claim, we need to argue that (1) $\mathcal{L}(M') \subseteq L^{\mathcal{R}}$ and that (2) $L^{\mathcal{R}} \subseteq \mathcal{L}(M')$.

To prove (1), consider an $x \in \mathcal{L}(M')$. By definition, this means that we can write $x = a_1 a_2 \ldots a_n$ with each $a_i \in \Sigma_\varepsilon$ and find a sequence $r_0, r_1, \ldots, r_n$ of states of M' such that

- $r_0 = q_{\text{new}}$, the start state of M',
- $r_i \in \delta'(r_{i-1}, a_i)$, for $1 \le i \le n$, and
- $r_n \in \{q_0\}$, the set of final states of M'.

By construction, a_1 must be ε, because otherwise the set $\delta'(r_0, a_1) = \delta'(q_{\text{new}}, a_1)$ would be empty. Also, the states r_{i-1} for $2 \le i \le n$ must all be different from q_{new}, and so, every a_i for $2 \le i \le n$ must be different from ε (i.e., in the set Σ). Therefore, by construction, $\delta'(r_{i-1}, a_i)$ for $2 \le i \le n$ is the set of all r such that $\delta(r, a) = r_{i-1}$. Since, by the second bullet above, r_i is in this set, it follows that $\delta(r_i, a_i) = r_{i-1}$. Finally, the state r_1 must lie in F, because it must lie in $\delta'(r_0, a_1) = \delta'(q_{\text{new}}, \varepsilon) = F$. Thus, we have

- $r_n = q_0$, the start state of M,
- $r_{i-1} = \delta(r_i, a_i)$, for $n \ge i \ge 2$, and
- $r_1 \in F$, the set of final states of M.

Thus we have a sequence $r_n, r_{n-1}, \dots, r_1$ of states of M which satisfies the above three bullets; this ensures that $a_n a_{n-1} \dots a_2 \in \mathcal{L}(M) = L$, i.e., that $x^{\mathcal{R}} \in L$, i.e., that $x \in L^{\mathcal{R}}$.

The proof of (2) is very similar, so we only sketch it here. We start with an $x \in L^{\mathcal{R}}$. This means $x^{\mathcal{R}} \in L = \mathcal{L}(M)$. Therefore $x^{\mathcal{R}}$ takes M through a sequence $r_0, r_1, \dots, r_n$ of steps with $r_0 = q_0$ and $r_n \in F$. Arguing just as above, we can show that x can take M' through the sequence of states $q_{\text{new}}, r_n, r_{n-1}, \dots, r_0$ and since q_{new} is the start state of M' and r_0 is an accept state of M', we see that M' accepts x. So $x \in \mathcal{L}(M')$.

4. Let L_1, L_2 be two languages over the same alphabet Σ. Prove that $(L_1^* L_2^*)^* = (L_1 \cup L_2)^*$. Remember that to prove $A = B$ for sets A and B you must separately prove the two statements $A \subseteq B$ and $B \subseteq A$.

Solution: Suppose $x \in (L_1^* L_2^*)^*$. By definition of Kleene star, x is a concatenation of zero or more strings, each in $L_1^* L_2^*$, i.e. $x = x_1 x_2 \dots x_n$ with each $x_i \in L_1^* L_2^*$. Again, by definition, each $x_i = y_{i1} y_{i2} \dots y_{is_i} z_{i1} z_{i2} \dots z_{it_i}$ with each $y_{ij} \in L_1$ and each $z_{ij} \in L_2$. Putting it all together:

$$x = y_{11} \dots y_{1s_1} z_{11} \dots z_{1t_1} y_{21} \dots y_{2s_2} z_{21} \dots z_{2t_2} \dots\dots\dots z_{nt_n} .$$

Since each y_{ij} and each z_{ij} is in $L_1 \cup L_2$, it follows that $x \in (L_1 \cup L_2)^*$. We have shown that $(L_1^* L_2^*)^* \subseteq (L_1 \cup L_2)^*$.

Now, suppose $x \in (L_1 \cup L_2)^*$. Then $x = x_1 x_2 \dots x_n$ where each $x_i \in L_1 \cup L_2$. If an $x_i \in L_1$, we can write $x_i = x_i \varepsilon$ which puts it in $L_1^* L_2^*$. If an $x_i \in L_2$, we can similarly write $x_i = \varepsilon x_i$ which puts it in $L_1^* L_2^*$. Therefore, each $x_i \in L_1^* L_2^*$, whence $x \in (L_1^* L_2^*)^*$. We have shown that $(L_1 \cup L_2)^* \subseteq (L_1^* L_2^*)^*$.

This completes the proof.

Nicolas Baum's Solution: Since $L_1^* \subseteq (L_1 \cup L_2)^*$ and $L_2^* \subseteq (L_1 \cup L_2)^*$, we have $L_1^* L_2^* \subseteq (L_1 \cup L_2)^* (L_1 \cup L_2)^* = (L_1 \cup L_2)^*$. Taking a Kleene star on both sides, we get $(L_1^* L_2^*)^* \subseteq ((L_1 \cup L_2)^*)^* = (L_1 \cup L_2)^*$.

Since $L_1 \subseteq L_1^* \subseteq L_1^* L_2^*$ and $L_2 \subseteq L_2^* \subseteq L_1^* L_2^*$, we have $L_1 \cup L_2 \subseteq L_1^* L_2^*$. Taking a Kleene star on both sides, we get $(L_1 \cup L_2)^* \subseteq (L_1^* L_2^*)^*$.

5. Prove that there exist languages $A, B, C \subseteq \{0,1\}^*$ that satisfy all of the following properties:

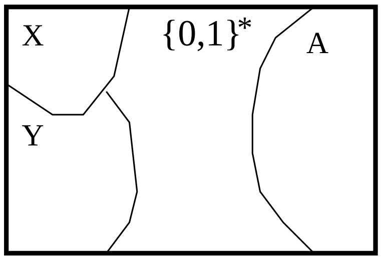
(a) $A = B \cap C$.

(b) B and C are both non-regular.

(c) A is infinite and regular.

To get any credit, you *must* prove *all* three properties for whatever A, B, C you have decided to use.

Solution: Define the sets $X = \{0^{n^2} : n \geq 0\}$, $Y = 0^* - X$ and $A = 11^*$. Note that, by construction, any two of X, Y and A are *disjoint*; so a Venn diagram of these three sets would look like this:



Define $B = X \cup A$, $C = Y \cup A$. By the disjointness observed above (see diagram), condition (a) clearly holds. Since $A = 11^*$ is regular, condition (c) also holds. The only nontrivial thing is to establish condition (b). Let us use bars to denote complements of languages with respect to $\{0,1\}^*$. Then, by the disjointness conditions (see diagram), $X = B - A = B \cap \overline{A}$. Similarly, $Y = C \cap \overline{A}$. Now we shall establish condition (b) using a proof by contradiction.

First, suppose B is regular. Then, by closure of regular languages under complement and intersection, $X = B \cap \overline{A}$ would also be regular. But we've prove in class that X is not regular, so we have a contradiction. Thus, B must be non-regular. Next, suppose C is regular. Again, by closure properties, $Y = C \cap \overline{A}$ would be regular. The complement of Y *with respect to* 0^* is X, so X must also be regular and again we have a contradiction. Thus, C must be non-regular.

6. A permutation of a string x is any string that can be obtained by rearranging the characters of x. Thus, for example, the string abc has exactly six permutations:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba\,.$$

Clearly, if y is a permutation of x, then $|y| = |x|$. For a language L over alphabet Σ, define

$$\begin{aligned} \textsc{Permute}(L) &= \{x \in \Sigma^* : x \text{ is a permutation of some string in } L\}\,, \\ \textsc{Select}(L) &= \{x \in \Sigma^* : \text{every permutation of } x \text{ is in } L\}\,. \end{aligned}$$

Classify each of the following statements as TRUE or FALSE, and give proofs justifying your classifications.

6.1. If $L_1 = 1^*0$, then $\textsc{Permute}(L_1)$ is regular.
Solution: TRUE. In this case $\textsc{Permute}(L_1) = \{x \in \{0,1\}^* : x \text{ contains exactly one } 0\} = 1^*01^*$, which is clearly regular.

6.2. If $L_1 = 1^*0$, then $\textsc{Select}(L_1)$ is regular.
Solution: TRUE. Any string in $\textsc{Select}(L_1)$ must clearly be in L_1 (since a string is always a permutation of itself); however, a string in L_1 must end in a 0, so permuting it will destroy this property unless the string is just a plain 0. Thus, $\textsc{Select}(L_1) = \{0\}$, which is clearly regular.

6.3. Regular languages are closed under the operation PERMUTE.
Solution: FALSE. For a counterexample, consider $L = (01)^*$. Any string in L contains an equal number of 0's and 1's, so, when we take all permutations of all strings in L, we get precisely the language $\{x \in \{0,1\}^* : x \text{ contains as many 0's as 1's}\}$. We have proved in class, using the pumping lemma, that this language is not regular. Thus $\textsc{Permute}(L)$ need not be regular even if L is.

6.4. Regular languages are closed under the operation SELECT.

Solution: FALSE. For a counterexample, consider $L' = \overline{(01)^*}$, with the complement being taken with respect to alphabet $\{0,1\}$. Clearly L' is regular. If a string x has an unequal number of 0's and 1's, then no matter how we permute it we will never land in the set $(01)^*$, i.e., every permutation of x is in L', i.e., $x \in \textsc{Select}(L')$. On the other hand, if a string x has as many 0's as 1's, then it *can* be permuted into the form $(01)^*$, so $x \notin \textsc{Select}(L')$. In short, we have shown that $\textsc{Select}(L') = \{x \in \{0,1\}^* : x$ contains an unequal number of 0's and 1's$\}$. Thus, $\textsc{Select}(L')$ is the complement of a non-regular language, whence it is itself non-regular.

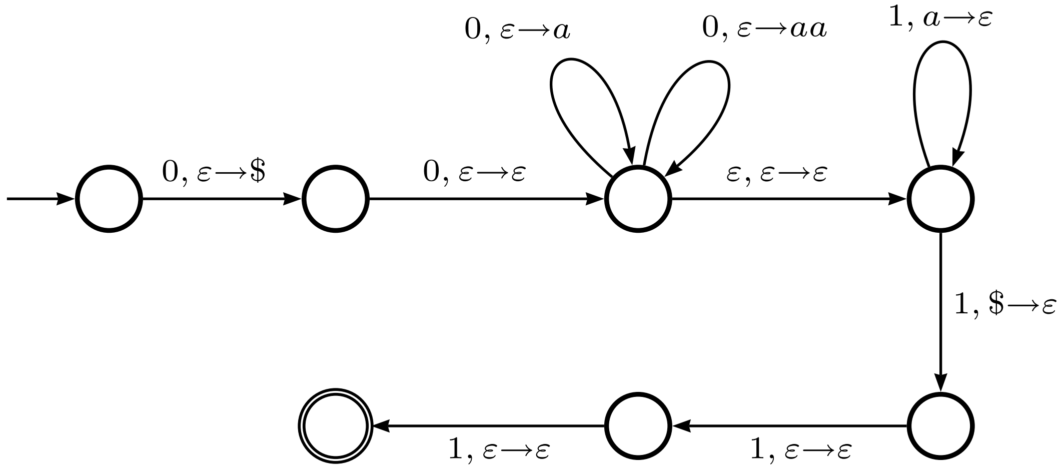
7. Draw a PDA for the language $\{0^i 1^j : i < j < 2i\}$. For clarity, keep your stack alphabet disjoint from $\{0,1\}$. Provide a brief justification (no need for a formal proof) that your PDA works correctly.

Solution: Let L be the given language. First, let us try to build a PDA for a slightly easier language: $L' = \{0^i 1^j : i \le j \le 2i\}$. The idea is to push either one or two a's (choose nondeterministically) onto our stack as we read the 0's in the input string, and then pop one symbol at a time as we read the 1's. If we empty the stack at the same time as when we finish reading the input, we may accept.

But for the language L, we must reject strings of the form $0^i 1^i$ and $0^i 1^{2i}$. To do so, observe that the shortest string in L is 00111 and that any string in L is therefore of the form

$$000^{i'} 1^{j'} 111$$

for some $i' \ge 0, j' \ge 0$, and $(i'+2) < (j'+3) < 2(i'+2)$. Since i' and j' are integers, the latter condition is equivalent to $i' \le j' \le 2i'$. Thus, strings in L are simply strings in L' with two 0's prepended and three 1's appended. With this in mind, we build our PDA:



8. Design a context-free grammar for the *complement* of the language $\{a^n b^n : n \ge 0\}$ over the alphabet $\{a,b\}$. Give brief explanations for the "meanings" of your variables (i.e. explain what strings are generated by each of your variable).

Solution: A string in the complement of $\{a^n b^n : n \ge 0\}$ is either of the form $\{a^i b^j : i > j \ge 0\}$ or of the form $\{a^i b^j : j > i \ge 0\}$ or not of the form $a^* b^*$, which means that it contains the substring ba. In the following grammar, the variable U generates strings of the third type, while T generates strings of the form $a^n b^n$ to which we either prepend a string of one or more a's (generated by A) or append a string of one or more b's (generated by B).

$$\begin{aligned}
S &\longrightarrow AT \mid TB \mid U \\
A &\longrightarrow Aa \mid a \\
B &\longrightarrow Bb \mid b \\
T &\longrightarrow aTb \mid \varepsilon \\
U &\longrightarrow VbaV \\
V &\longrightarrow Va \mid Vb \mid \varepsilon
\end{aligned}$$

Jason Reeves's Solution: The following CFG works (proof left as an exercise to the reader):

$$\begin{aligned}
A &\longrightarrow C \mid aAb \\
B &\longrightarrow C \mid \varepsilon \mid aBb \\
C &\longrightarrow a \mid b \mid aBa \mid bBb \mid bBa
\end{aligned}$$