Solutions to Quiz 1
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1. Give a *one sentence* proof that any NFA can be converted into an equivalent NFA that has no ε-transitions. Feel free to use results proved in class.

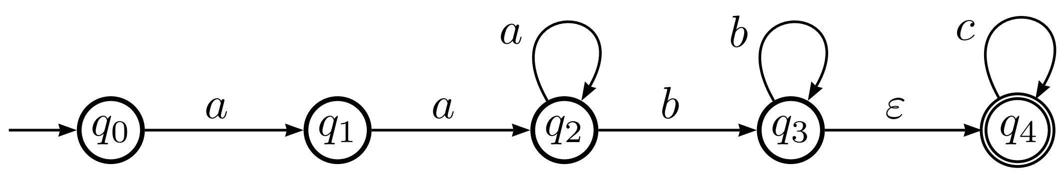
   [10 points]

   Solution: As proved in class, any NFA can be converted into an equivalent DFA, which is, of course, automatically an NFA without ε-transitions.

2. Draw an NFA that recognizes the language $\{a^i b^j c^k : i \geq 2, j \geq 1, k \geq 0\}$. Keep it simple!

   [10 points]

   Solution:



3. Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be two DFAs over the same alphabet Σ. Write a formal description of a DFA that recognizes the language $\mathcal{L}(M_1) \cap \mathcal{L}(M_2)$. No proof of correctness required.

[10 points]

Solution: The following DFA, M, recognizes $\mathcal{L}(M_1) \cap \mathcal{L}(M_2)$.

$$M \;=\; (Q_1 \times Q_2,\ \Sigma,\ \delta,\ (q_1, q_2),\ F_1 \times F_2)\,,$$

where δ is given by

$$\delta((q, r), a) \;=\; (\delta_1(q, a), \delta_2(r, a)), \quad \forall\, q \in Q_1, r \in Q_2, a \in \Sigma\,.$$

4. Write a regular expression for the language $\{x \in \{R, G, B\}^* : x$ has an odd number of R's$\}$. You do not have to give a proof of correctness.

[10 points]

Solution: The intuition is that any string in the given language consists of

- an initial block of symbols ending in the string's first R, followed by
- zero or more blocks, each containing two R's and ending in the second R, followed by
- a final block of symbols containing no R's.

Thus, the following regular expression generates the language:

$$(G \cup B)^* R \left((G \cup B)^* R (G \cup B)^* R\right)^* (G \cup B)^*\,.$$

5. In class, we went through a proof that a DFA $M = (\{q_1, \ldots, q_n\}, \Sigma, \delta, q_1, F)$ can be converted into an equivalent regular expression. During this proof we defined some sets R_{ij}^k and proved, using induction, that they were all regular.

5.1. Define the sets R_{ij}^k. An informal definition will suffice.

[5 points]

Solution: R_{ij}^k is the set of all strings in Σ^* that take M from state q_i to state q_j without ever passing through a state numbered higher than q_k. Here, "passing through" means entering and then leaving.

5.2. Write down a system of equations which inductively, and completely, specifies every set R_{ij}^k. Make sure your equations cover all the base cases and all the induction steps. You do not need to prove that your equations are correct.

[10 points]

Solution: Here are the desired equations:

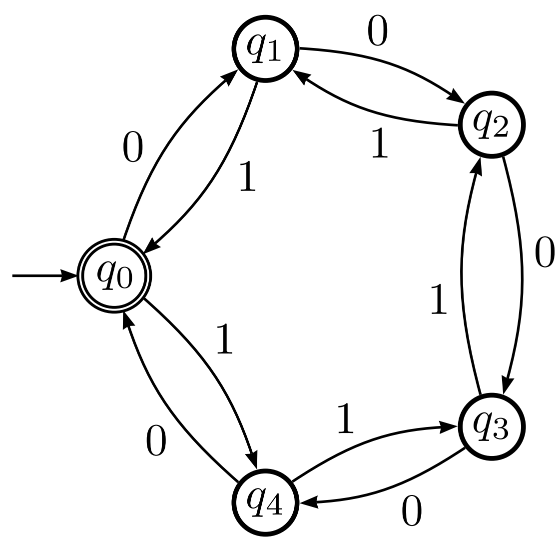
$$\begin{aligned} R_{ij}^0 &= \{a \in \Sigma : \delta(q_i, a) = q_j\}, && \text{for } i, j \in \{1, \ldots, n\} \text{ with } i \neq j\,, \\ R_{ii}^0 &= \{\varepsilon\} \cup \{a \in \Sigma : \delta(q_i, a) = q_i\}, && \text{for } i \in \{1, \ldots, n\}\,, \\ R_{ij}^k &= R_{ij}^{k-1} \cup R_{ik}^{k-1} \left(R_{kk}^{k-1}\right)^* R_{kj}^{k-1}, && \text{for } i, j, k \in \{1, \ldots, n\}\,. \end{aligned}$$

6. For a string $x \in \{0,1\}^*$, let $N_0(x)$ and $N_1(x)$ denote the number of 0's and 1's (respectively) in x. Of the following two languages, exactly one is regular. Specify which language is regular and prove that it is so. You do not have to give a proof that the other language is not regular.

$$\begin{aligned} L_1 &= \{x \in \{0,1\}^* : |N_0(x) - N_1(x)| = 5\}\,, \\ L_2 &= \{x \in \{0,1\}^* : |N_0(x) - N_1(x)| \text{ is divisible by } 5\}\,. \end{aligned}$$

[15 points]

Solution: The language L_2 is regular, because it is recognized by the following DFA.



The language L_1 is not regular. You did not have to prove this, but you should be able to do so by using the pumping lemma to try to pump the string $0^{p+5}1^p \in L_1$, where p is the hypothetical pumping length of L_1.