Homework 3 Solutions

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1. **(DFA → regular expression)**

1.1. $R^0_{11} = \varepsilon \cup a$
$R^0_{12} = b$
$R^0_{21} = b$
$R^0_{22} = \varepsilon \cup a$

$R^1_{11} = R^0_{11} \cup R^0_{11}(R^0_{11})^* R^0_{11} = (\varepsilon \cup a)^+ = a^*$
$R^1_{12} = R^0_{12} \cup R^0_{11}(R^0_{11})^* R^0_{12} = b \cup (\varepsilon \cup a)^+ b = a^* b$
$R^1_{21} = R^0_{21} \cup R^0_{21}(R^0_{11})^* R^0_{11} = b \cup b(\varepsilon \cup a)^+ = ba^*$
$R^1_{22} = R^0_{22} \cup R^0_{21}(R^0_{11})^* R^0_{12} = (\varepsilon \cup a) \cup b(\varepsilon \cup a)^* b = \varepsilon \cup a \cup ba^* b$

$R^2_{12} = R^1_{12} \cup R^1_{12}(R^1_{22})^* R^1_{22} = a^* b \cup a^* b(\varepsilon \cup a \cup ba^* b)^+ = a^* b(a \cup ba^* b)^*$
$\Rightarrow L = R^2_{12} = a^* b(a \cup ba^* b)^*$

1.2. $R^0_{11} = \varepsilon$
$R^0_{12} = a \cup b$
$R^0_{13} = \phi$
$R^0_{21} = \phi$
$R^0_{22} = \varepsilon \cup a$
$R^0_{23} = b$
$R^0_{31} = a$
$R^0_{32} = b$
$R^0_{33} = \varepsilon$

$R^1_{11} = \varepsilon$
$R^1_{12} = a \cup b$
$R^1_{13} = \phi$
$R^1_{21} = \phi$
$R^1_{22} = \varepsilon \cup a$
$R^1_{23} = b$
$R^1_{31} = a$
$R^1_{32} = b \cup a(a \cup b) = b \cup aa \cup ab$
$R^1_{33} = \varepsilon$

$R^2_{11} = \varepsilon$
$R^2_{12} = a \cup b \cup (a \cup b)a^* = a^+ \cup ba^*$
$R^2_{13} = (a \cup b)a^* b = a^+ b \cup ba^* b$
$R^2_{21} = \phi$
$R^2_{22} = a^*$
$R^2_{23} = b \cup (\varepsilon \cup a)a^* b = a^* b$
$R^2_{31} = a$
$R^2_{32} = (b \cup aa \cup ab) \cup (b \cup aa \cup ab)a^*(\varepsilon \cup a) = ba^* \cup aa^+ \cup aba^*$
$R^2_{33} = \varepsilon \cup (b \cup aa \cup ab)a^* b = \varepsilon \cup ba^* b \cup aa^+ b \cup aba^* b$

$R^3_{11} = \varepsilon \cup (a^+ b \cup ba^* b)(ba^* b \cup aa^+ b \cup aba^* b)^* a$
$R^3_{13} = (a^+ b \cup ba^* b)(ba^* b \cup aa^+ b \cup aba^* b)^*$
$\Rightarrow L = R^3_{11} \cup R^3_{13}$

2. **(True or false)**

   2.1. False. For a counterexample, let L be any non-regular language such as $\{0^n1^n : n \geq 0\}$. Since L is non-regular, $\overline{L}$ (the complement of L) is also nonregular. Yet, $L \cup \overline{L} = \Sigma^*$ is regular.

   2.2. False. For a counterexample, let L be any non-regular language such as $\{0^n1^n : n \geq 0\}$. Since L is non-regular, $\overline{L}$ (the complement of L) is also nonregular. Yet, $L \cap \overline{L} = \Phi$ is regular.

   2.3. True. We proved in class that if a language is regular, so is its complement. This is equivalent to the statement that if the complement of a language is regular, so is that language itself.

   2.4. False. Any set can be written as a (possibly infinite) union of singleton sets containing its elements. In particular, any language L can be written as a union of finite, therefore regular, languages: $L = \bigcup_{x \in L}\{x\}$. More concretely, take our favorite nonregular language. We have $\{0^n1^n : n \geq 0\} = \bigcup_{n=0}^{\infty}\{0^n1^n\}$.

   2.5. False. If this were true, then by De Morgan's Law the previous would also have to be true. For a concrete counterexample, let $A_n = \{0^n1^n\}$ for every $n \geq 0$. Then for every n, $\overline{A_n}$ is regular. Assume, to get a contradiction, that the statement is true. Then $\bigcap_{n=0}^{\infty}\overline{A_n}$ is regular, so that $\overline{\bigcap_{n=0}^{\infty}\overline{A_n}}$ is regular. But by De Morgan's Law, the latter is just $\bigcup_{n=0}^{\infty} A_n$, which we know to be nonregular, giving us our contradiction.

3. **(L regular $\Rightarrow$ $\textsc{Max}(L)$ regular)**

   If $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA for L, then the intuition to construct a DFA M' for $\textsc{Max}(L)$ is as follows. If q_f is a final state of M and there is a non-empty string that drives M from q_f to a final state (possibly q_f itself), then q_f should not be a final state in M'. This ensures that M' does not accept a string in L if there is a way of extending it to be another string in L.

   Formally, let $M' = (Q, \Sigma, \delta, q_0, F')$, where $F' = \{q : q \in F \text{ and } \forall x \in \Sigma^+, \hat{\delta}(q, x) \notin F\}$.

4. **(L regular $\Rightarrow$ $\textsc{Cycle}(L)$ regular)**

   We observe that a string w is in $\textsc{Cycle}(L)$ if and only if there is a way to split w into two parts: x_1 and x_2, such that there is a state q of L's DFA M satisfying

   1. $\hat{\delta}(q, x_1) \in F$ and
   2. $\hat{\delta}(q_0, x_2) = q$.

   That is to say, a marble starting off in state q ends up in a final state of M upon consuming x_1, and a marble starting off in the initial state of M ends up in q upon consuming x_2. This suggests that the marble should keep track of three things: (1) the state of M where it started, (2) the state of M at which it currently is, and (3) whether it is consuming x_1 or x_2. Accordingly, each state of the new NFA M' will be a 3-vector (p, q, i), where p and q are states of M, and $i \in \{1, 2\}$.

   Formally, if $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA for L, define a new NFA $M' = (Q', \Sigma, \delta', q_0', F')$, where

$$\begin{aligned}
Q' &= Q \times Q \times \{1,2\} \cup \{q_0'\} \text{ where } q_0' \notin Q \times Q \times \{1,2\} \\
F' &= \{(q, q, 2) : q \in Q\} \\
\delta'(q_0', \varepsilon) &= \{(q, q, 1) : q \in Q\} \\
\delta'((p, q, 1), \varepsilon) &= \{(p, q_0, 2)\} \text{ for every } q \in F \\
\delta'((p, q, i), a) &= \{(p, \delta(q, a), i)\} \text{ if } a \in \Sigma
\end{aligned}$$

   By the discussion above, M' recognizes $\textsc{Cycle}(L)$.

5. **(Regular or not?)**

5.1. $L = \{0^m 1^n 0^{m+n} : m, n \geq 0\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 1^p 0^p$, where p is the pumping length. Clearly, $s \in L$ (for $m = 0$ and $n = p$) and $|s| \geq p$, so let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| \leq p$, y must lie entirely within the sequence of 1's. Hence, $xz = 1^{p-|y|}0^p$ should belong to L by the Lemma, but it does not since $p - |y| \neq p$, giving us our contradiction.

5.2. $L = \{0^m 1^n : m \text{ divides } n\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 0^p 1^p$, where p is the pumping length. Again, let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| \leq p$, y must lie entirely within the sequence of 0's. Hence, $xy^2z = 0^{p+|y|}1^p$ should belong to L by the Lemma, but does not since $p + |y| > p$ so it certainly does not divide p, giving us our contradiction.

5.3. $L = \{xwx^R : x, w \in \{0,1\}^* \text{ and } |x|, |w| > 0\}$.
Regular. Careful observation will reveal that a string is in L if and only if it starts and ends with the same symbol and is of length at least three. L is therefore captured by the regular expression $0(0 \cup 1)^+0 \cup 1(0 \cup 1)^+1$.

5.4. $L = \{0^{2^n} : n \geq 0\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = 0^{2^p}$, where p is the pumping length. Let $s = xyz$ as specified by the Pumping Lemma. Then by the Lemma, $xy^2z \in L$. However, clearly $|xy^2z| > |xyz| = 2^p$, yet $|xy^2z| < 2^{p+1}$ since $|y| \leq |xy| \leq p < 2^p$, so $xy^2z \notin L$, giving us our contradiction.

5.5. $L = \{w \in \Sigma_2^* : \text{the bottom row of } w \text{ is the reverse of the top row of } w\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let $s = \begin{bmatrix}0\\0\end{bmatrix}^p \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}^p$, where p is the pumping length. Let $s = xyz$ as specified by the Pumping Lemma. Since $|xy| < p$, y lies entirely in the first sequence of $\begin{bmatrix}0\\0\end{bmatrix}$'s. Hence, $xz = \begin{bmatrix}0\\0\end{bmatrix}^{p-|y|} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\0\end{bmatrix}^p$ does not belong to L, contradicting the Lemma.

5.6. $L = \{0^m 1^n : m, n \geq 0 \text{ and } m \neq n\}$.
Nonregular. Assume, to get a contradiction, that L is regular. Let R denote the language captured by the regular expression 0^*1^*. Then $L \cup \overline{R}$, and therefore $\overline{L \cup \overline{R}}$, must be regular since the set of all regular languages is closed under union and complementation. But the latter expression is precisely the language $\{0^n 1^n : n \geq 0\}$, which we know to be nonregular, giving us our contradiction.