Solutions to Mid-Term Exam
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Who Graded What

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1. Write a regular expression for the language generated by the following grammar:

$$\begin{aligned} S &\longrightarrow AT \\ T &\longrightarrow ABT \mid TBA \mid AA \\ A &\longrightarrow 0 \\ B &\longrightarrow 1 \end{aligned}$$

Your regular expression should be as simple as possible. No proof of correctness required.

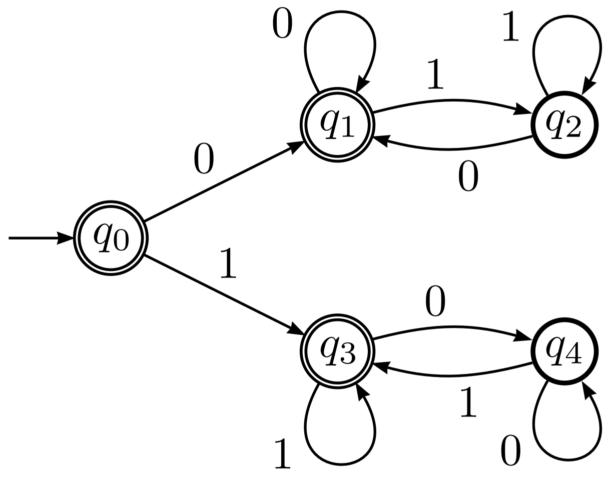
Solution: The grammar generates $0(01)^*00(10)^*$.

2. Draw a DFA (no proof required) for the language

$$\{x \in \{0,1\}^* : x \text{ contains an equal number of occurrences of the substrings } 01 \text{ and } 10\}.$$

For example, 101 and 0000 are in the language, but 1010 is not.

Solution: The idea is to handle strings beginning with a 1 and strings beginning with a 0 separately. The following DFA does the job:



3. This problem has two parts, each of which asks you to prove a closure property of regular languages. In each case, if your proof involves constructing a DFA/NFA, then you must (1) formally describe the machine you are constructing and (2) explain why your construction is correct. For step (2), an informal argument will suffice (though if you know how to write a formal proof, that is also welcome).

3.1. Recall that $x^{\mathcal{R}}$ denotes the reverse of the string x. For a language L, define

$$L^{\mathcal{R}} \;=\; \{x^{\mathcal{R}} : x \in L\}\,.$$

Prove that if L is regular, so is $L^{\mathcal{R}}$.

Solution: The idea is to start with a DFA for the regular language L and "reverse all the arrows" in this DFA, make its start state an accept state of the resulting machine, and make all its accept states start states (or rather, since only one start state is allowed, to introduce ε-transitions from a new start state to all the former accept states). This procedure clearly gives us an NFA, because, for instance, if there are five different states of the initial DFA that all lead into a state q on reading input symbol a, then after the conversion, q will lead to these five different states on input a. Now we give a formal proof.

Suppose L is a regular language. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing L. Let us construct an NFA $M' = (Q \cup \{q_{\text{new}}\}, \Sigma, \delta', q_{\text{new}}, \{q_0\})$ where δ' is given by

$$\begin{aligned}
\delta'(q_{\text{new}}, \varepsilon) &= F\,, & \\
\delta'(q_{\text{new}}, a) &= \emptyset\,, & \forall\, a \in \Sigma\,, \\
\delta'(q, \varepsilon) &= \emptyset\,, & \forall\, q \in Q\,, \\
\delta'(q, a) &= \{r \in Q :\; \delta(r, a) = q\}\,, & \forall\, q \in Q,\; a \in \Sigma\,.
\end{aligned}$$

We claim that $\mathcal{L}(M') = L^{\mathcal{R}}$, which would prove that $L^{\mathcal{R}}$ is regular. To prove our claim, we need to argue that (1) $\mathcal{L}(M') \subseteq L^{\mathcal{R}}$ and that (2) $L^{\mathcal{R}} \subseteq \mathcal{L}(M')$.

To prove (1), consider an $x \in \mathcal{L}(M')$. By definition, this means that we can write $x = a_1 a_2 \ldots a_n$ with each $a_i \in \Sigma_\varepsilon$ and find a sequence $r_0, r_1, \ldots, r_n$ of states of M' such that

- $r_0 = q_{\text{new}}$, the start state of M',
- $r_i \in \delta'(r_{i-1}, a_i)$, for $1 \le i \le n$, and
- $r_n \in \{q_0\}$, the set of final states of M'.

By construction, a_1 must be ε, because otherwise the set $\delta'(r_0, a_1) = \delta'(q_{\text{new}}, a_1)$ would be empty. Also, the states r_{i-1} for $2 \le i \le n$ must all be different from q_{new}, and so, every a_i for $2 \le i \le n$ must be different from ε (i.e., in the set Σ). Therefore, by construction, $\delta'(r_{i-1}, a_i)$ for $2 \le i \le n$ is the set of all r such that $\delta(r, a) = r_{i-1}$. Since, by the second bullet above, r_i is in this set, it follows that $\delta(r_i, a_i) = r_{i-1}$. Finally, the state r_1 must lie in F, because it must lie in $\delta'(r_0, a_1) = \delta'(q_{\text{new}}, \varepsilon) = F$. Thus, we have

- $r_n = q_0$, the start state of M,
- $r_{i-1} = \delta(r_i, a_i)$, for $n \ge i \ge 2$, and
- $r_1 \in F$, the set of final states of M.

Thus we have a sequence $r_n, r_{n-1}, \ldots, r_1$ of states of M which satisfies the above three bullets; this ensures that $a_n a_{n-1} \ldots a_2 \in \mathcal{L}(M) = L$, i.e., that $x^{\mathcal{R}} \in L$, i.e., that $x \in L^{\mathcal{R}}$.

The proof of (2) is very similar, so we only sketch it here. We start with an $x \in L^{\mathcal{R}}$. This means $x^{\mathcal{R}} \in L = \mathcal{L}(M)$. Therefore $x^{\mathcal{R}}$ takes M through a sequence $r_0, r_1, \ldots, r_n$ of steps with $r_0 = q_0$ and $r_n \in F$. Arguing just as above, we can show that x can take M' through the sequence of states $q_{\text{new}}, r_n, r_{n-1}, \ldots, r_0$ and since q_{new} is the start state of M' and r_0 is an accept state of M', we see that M' accepts x. So $x \in \mathcal{L}(M')$.

3.2. Fix an alphabet Σ. For strings $x = a_1 a_2 \cdots a_n$ and $y = b_1 b_2 \cdots b_n$ with the same length n, where each $a_i \in \Sigma$ and each $b_i \in \Sigma$, define the "perfect shuffle" of x and y — denoted $\textsc{Shuffle}(x, y)$ — to be the string $a_1 b_1 a_2 b_2 \cdots a_n b_n$. Thus, for example,

$$\textsc{Shuffle}(abc, dba) \;=\; adbbca, \quad \text{and} \quad \textsc{Shuffle}(20012, 01211) \;=\; 2001021121\,.$$

For languages $A, B \subseteq \Sigma^*$, define

$$\text{SHUFFLE}(A, B) \;=\; \{\text{SHUFFLE}(x, y) : |x| = |y|,\, x \in A \text{ and } y \in B\}\,.$$

Prove that if A and B are regular, so is $\text{SHUFFLE}(A, B)$.

Solution: Let $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ and $M_B = (Q_A, \Sigma, \delta_B, q_B, F_B)$ be DFAs recognizing A and B respectively. Let $C = \text{SHUFFLE}(A, B)$. To recognize C, we build a DFA whose states keep track of three pieces of information as we read an input string z:

- The state in Q_A reached by following M_A on the odd-position characters of z.
- The state in Q_B reached by following M_B on the even-position characters of z.
- A tag in $\{0, 1\}$, keeping track of the position of the last input character mod 2.

We start in state $(q_A, q_B, 0)$ and accept if we reach $(q, r, 0)$ for some $q \in F_A$ and $r \in F_B$. Now we formalize this construction.

The DFA to recognize C is $(Q, \Sigma, \delta, q_0, F)$, where

$$\begin{aligned} Q &= Q_A \times Q_B \times \{0,1\}\,, \\ q_0 &= (q_A, q_B, 0)\,, \\ F &= F_A \times F_B \times \{0\}\,, \end{aligned}$$

and δ is given by

$$\delta((q, r, t), a) \;=\; \begin{cases} (\delta_A(q, a), r, 1), & \text{if } t = 0, \\ (q, \delta_B(r, a), 0), & \text{if } t = 1. \end{cases}$$

The informal discussion above shows that this DFA works correctly.

4. Prove that there exist languages $A, B, C \subseteq \{0, 1\}^*$ that satisfy all of the following properties:

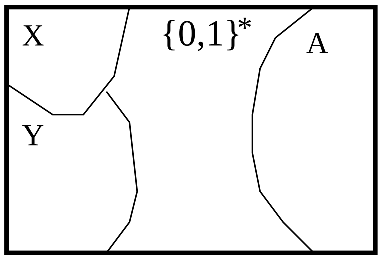
(a) $A = B \cap C$.

(b) B and C are both non-regular.

(c) A is infinite and regular.

To get any credit, the languages A, B, C you pick must satisfy *all three* properties. Further, you *must* prove the three properties for whatever A, B, C you have decided to use.

Solution: Define the sets $X = \{0^{n^2} : n \geq 0\}$, $Y = 0^* - X$ and $A = 11^*$. Note that, by construction, any two of X, Y and A are *disjoint*; so a Venn diagram of these three sets would look like this:



Define $B = X \cup A$, $C = Y \cup A$. By the disjointness observed above (see diagram), condition (a) clearly holds. Since $A = 11^*$ is regular, condition (c) also holds. The only nontrivial thing is to establish condition (b). Let us use bars to denote complements of languages with respect to $\{0, 1\}^*$. Then, by the disjointness conditions (see diagram), $X = B - A = B \cap \overline{A}$. Similarly, $Y = C \cap \overline{A}$. Now we shall establish condition (b) using a proof by contradiction.

First, suppose B is regular. Then, by closure of regular languages under complement and intersection, $X = B \cap \overline{A}$ would also be regular. But we've prove in class that X is not regular, so we have a contradiction. Thus, B must be non-regular. Next, suppose C is regular. Again, by closure properties, $Y = C \cap \overline{A}$ would be regular. The complement of Y *with respect to* 0^* is X, so X must also be regular and again we have a contradiction. Thus, C must be non-regular.

5. A permutation of a string x is any string that can be obtained by rearranging the characters of x. Thus, for example, the string abc has exactly six permutations:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba\,.$$

Clearly, if y is a permutation of x, then $|y| = |x|$. For a language L over alphabet Σ, define

$$\text{PERMUTE}(L) \quad = \quad \{x \in \Sigma^* : x \text{ is a permutation of some string in } L\}\,.$$

Are regular languages closed under the operation PERMUTE? Justify your answer with a formal proof.

Solution: No, regular languages are not closed under PERMUTE. For a counterexample, consider $L = (01)^*$. Any string in L contains an equal number of 0's and 1's, so, when we take all permutations of all strings in L, we get precisely the language $\{x \in \{0,1\}^* : x$ contains as many 0's as 1's$\}$. We have proved in class, using the pumping lemma, that this language is not regular. Thus PERMUTE(L) need not be regular even if L is.

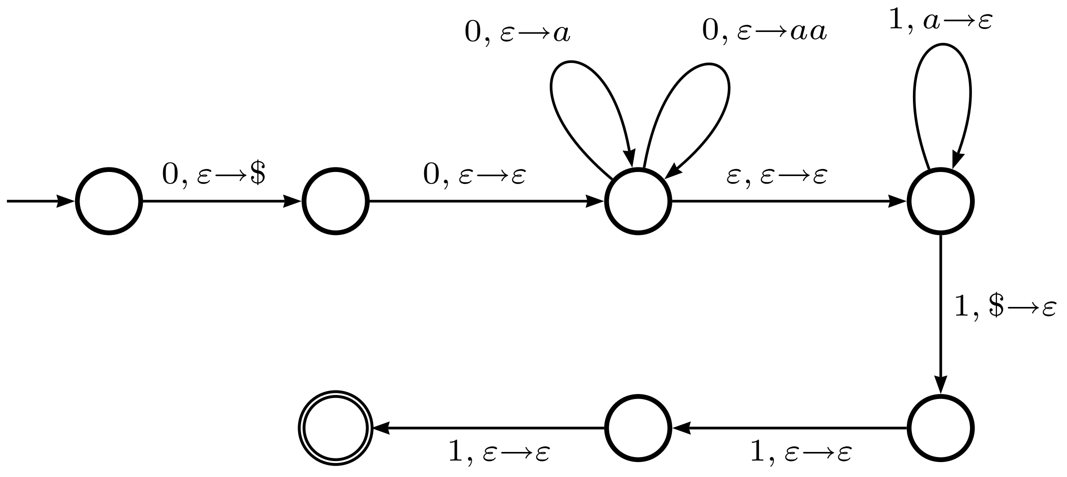
6. Draw a PDA for the language $\{0^i1^j : i < j < 2i\}$. Provide a brief justification (no need for a formal proof) that your PDA works correctly.

Solution: Let L be the given language. First, let us try to build a PDA for a slightly easier language: $L' = \{0^i1^j : i \le j \le 2i\}$. The idea is to push either one or two a's (choose nondeterministically) onto our stack as we read the 0's in the input string, and then pop one symbol at a time as we read the 1's. If we empty the stack at the same time as when we finish reading the input, we may accept.

But for the language L, we must reject strings of the form 0^i1^i and 0^i1^{2i}. To do so, observe that the shortest string in L is 00111 and that any string in L is therefore of the form

$$000^{i'}1^{j'}111$$

for some $i' \ge 0, j' \ge 0$, and $(i'+2) < (j'+3) < 2(i'+2)$. Since i' and j' are integers, the latter condition is equivalent to $i' \le j' \le 2i'$. Thus, strings in L are simply strings in L' with two 0's prepended and three 1's appended. With this in mind, we build our PDA:


$0, \varepsilon \rightarrow \$$
$0, \varepsilon \rightarrow \varepsilon$
$0, \varepsilon \rightarrow a$
$0, \varepsilon \rightarrow aa$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
$1, a \rightarrow \varepsilon$
$1, \$ \rightarrow \varepsilon$
$1, \varepsilon \rightarrow \varepsilon$
$1, \varepsilon \rightarrow \varepsilon$

7. Design a context-free grammar for the *complement* of the language $\{a^n b^n : n \geq 0\}$ over the alphabet $\{a, b\}$. Give brief explanations for the "meanings" of your variables (i.e., explain what strings are generated by each of your variables). No further proof is necessary.

Solution: A string in the complement of $\{a^n b^n : n \geq 0\}$ is either of the form $\{a^i b^j : i > j \geq 0\}$ or of the form $\{a^i b^j : j > i \geq 0\}$ or not of the form $a^* b^*$, which means that it contains the substring ba. In the following grammar, the variable U generates strings of the third type, while T generates strings of the form $a^n b^n$ to which we either prepend a string of one or more a's (generated by A) or append a string of one or more b's (generated by B).

$$\begin{array}{rcl} S & \longrightarrow & AT \mid TB \mid U \\ A & \longrightarrow & Aa \mid a \\ B & \longrightarrow & Bb \mid b \\ T & \longrightarrow & aTb \mid \varepsilon \\ U & \longrightarrow & VbaV \\ V & \longrightarrow & Va \mid Vb \mid \varepsilon \end{array}$$

Another Especially Elegant Solution: The following CFG, proposed by Ruslan Dimov '08, also works (think why).

$$S \longrightarrow aSb \mid aSa \mid bSb \mid bS \mid Sa \mid a \mid b$$