Mid-Term Exam
Due Nov 5, 2007

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Submission Instructions

- Submit this exam exactly as you would submit a regular homework, i.e., by 12:30pm sharp on the due date. Late penalties will be assessed exactly as for the homeworks.
- Please write or type your solutions neatly and **staple together your sheets of paper**. We are not responsible for sheets lost due to lack of stapling.

Honor Code

- **You must work on the exam alone**; you may not collaborate with anyone.
- You may consult your textbook (Sipser), your notes, and anything posted on the CS 39 website. Consulting anything else (e.g. last year's notes, your friend's notes, other websites) is a violation of the Honor Code.
- You may speak to others about the exam only in *complete generality* (e.g., "The exam is hard", "I'm almost finished with the exam", "I'll be working on the exam tonight"). You may not speak about the exam in any detail whatsoever (e.g., "Problem 3 is hard", "Problem 5 is easy", "That pumping lemma problem is tough").
- Since this is an exam, we (the TAs and the instructor) cannot help you with the particular problems on it, nor can we check if you are on the right track with a problem. However, as you attempt to solve these problems, if you discover that your understanding is not complete on some topics, please see any of us. We are willing to help you with your understanding of the course material to any degree.

Important General Instructions

- Every problem contains clear instructions about the level of justification/proof expected. When a formal proof is asked for, an answer containing only intuition (without the accompanying mathematics) will get very little credit.
- You may use, without proof, any results proven in class, in a homework, or in the textbook; simply cite the result you are using. Thus, for example, if you need the result "$\{0^n1^n : n \geq 0\}$ is a non-regular language" as part of some formal proof you are writing, you may simply cite the appropriate theorem in the textbook.
- Please read each question carefully. Unfortunately, if you misread and answer a different question than the one asked, you will not get credit.
- **Good luck!**

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1. Write a regular expression for the language generated by the following grammar:

$$\begin{aligned} S &\longrightarrow AT \\ T &\longrightarrow ABT \mid TBA \mid AA \\ A &\longrightarrow 0 \\ B &\longrightarrow 1 \end{aligned}$$

Your regular expression should be as simple as possible. No proof of correctness required.
[5 points]

2. Draw a DFA (no proof required) for the language

$$\{x \in \{0,1\}^* : x \text{ contains an equal number of occurrences of the substrings } 01 \text{ and } 10\}\,.$$

For example, 101 and 0000 are in the language, but 1010 is not.

[5 points]

3. This problem has two parts, each of which asks you to prove a closure property of regular languages. In each case, if your proof involves constructing a DFA/NFA, then you must (1) formally describe the machine you are constructing and (2) explain why your construction is correct. For step (2), an informal argument will suffice (though if you know how to write a formal proof, that is also welcome).

3.1. Recall that $x^{\mathcal{R}}$ denotes the reverse of the string x. For a language L, define

$$L^{\mathcal{R}} \;=\; \{x^{\mathcal{R}} : x \in L\}\,.$$

Prove that if L is regular, so is $L^{\mathcal{R}}$.

[10 points]

3.2. Fix an alphabet Σ. For strings $x = a_1a_2\cdots a_n$ and $y = b_1b_2\cdots b_n$ with the same length n, where each $a_i \in \Sigma$ and each $b_i \in \Sigma$, define the "perfect shuffle" of x and y — denoted $\textsc{Shuffle}(x, y)$ — to be the string $a_1b_1a_2b_2\cdots a_nb_n$. Thus, for example,

$$\textsc{Shuffle}(abc, dba) \;=\; adbbca, \quad \text{and} \quad \textsc{Shuffle}(20012, 01211) \;=\; 2001021121\,.$$

For languages $A, B \subseteq \Sigma^*$, define

$$\textsc{Shuffle}(A, B) \;=\; \{\textsc{Shuffle}(x, y) : |x| = |y|,\ x \in A \text{ and } y \in B\}\,.$$

Prove that if A and B are regular, so is $\textsc{Shuffle}(A, B)$.

[15 points]

4. Prove that there exist languages $A, B, C \subseteq \{0,1\}^*$ that satisfy all of the following properties:

(a) $A = B \cap C$.
(b) B and C are both non-regular.
(c) A is infinite and regular.

To get any credit, the languages A, B, C you pick must satisfy *all three* properties. Further, you *must* prove the three properties for whatever A, B, C you have decided to use.

[15 points]

5. A permutation of a string x is any string that can be obtained by rearranging the characters of x. Thus, for example, the string abc has exactly six permutations:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba\,.$$

Clearly, if y is a permutation of x, then $|y| = |x|$. For a language L over alphabet Σ, define

$$\textsc{Permute}(L) \quad = \quad \{x \in \Sigma^* : x \text{ is a permutation of some string in } L\}\,.$$

Are regular languages closed under the operation $\textsc{Permute}$? Justify your answer with a formal proof.

[10 points]

6. Draw a PDA for the language $\{0^i 1^j : i < j < 2i\}$. Provide a brief justification (no need for a formal proof) that your PDA works correctly.

[10 points]

7. Design a context-free grammar for the *complement* of the language $\{a^n b^n : n \geq 0\}$ over the alphabet $\{a, b\}$. Give brief explanations for the "meanings" of your variables (i.e., explain what strings are generated by each of your variables). No further proof is necessary.

[10 points]

Here endeth the exam.