Final Exam
March 2005
Attach this sheet to your submission

Prof. Amit Chakrabarti
Computer Science Department
Dartmouth College

I started this exam at time ______________ on date ______________.
I am submitting this exam at time ______________ on date ______________.

I pledge my honor that the times and dates I have reported above are accurate to the best of my knowledge.

Your signature:

Your name:

Important Points to Note

- This exam is due **48 hours** from when you first look at it, or on **March 15, at 6:00pm sharp**, whichever comes earlier. Submit it into Chakrabarti's mailbox.
- There are **6 sections** for a total of **100 points**.
- Start each problem (or part thereof) on a fresh page. Use **at most ten pages** (that's five sheets, letter or A4 size) for the entire exam. You will be held to this limit, strictly.
- The exam is open book and open notes. You may use, without proof, any results proved in class or in Sipser's book, provided you state clearly what result(s) you are using.
- You may not discuss this exam in any detail whatsoever with anyone, even after you have submitted it. If you have questions on the course's material, please ask the professor or the TA; we are willing to help you to any extent with understanding of the material.
- If you are completely stuck on a problem, **you may ask for a hint** which will be provided via email. Your score for that problem will then be halved. To ask for a hint, send email to Amit Chakrabarti (`ac@cs`) **as well as** Chien-Chung Huang (`villars@cs`) saying something like "I'd like the hint for Problem 5.1 and Problem 3." We will try to get back to you ASAP but please remember that we are humans too and we must sleep.
- Good luck!

Section	Points	Score
1	15	
2	10	
3	20	
4	10	
5	30	
6	15	
Total	**100**	

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1. For a string $x \in \{0,1\}^*$, let $\beta(x)$ denote the value of x when interpreted as a binary number. Thus, $\beta(1) = 1$, $\beta(100) = 4$, $\beta(001011) = 11$, and so on. Prove or disprove that the language $\{x \in \{0,1\}^* : \beta(x) \text{ is a perfect square}\}$ is regular.

   [15 points]

2. For a string $x \in \{0,1\}^*$, let $N_0(x)$ and $N_1(x)$ denote the number of 0's and 1's (respectively) in the string x. Prove or disprove that the language $\{x \in \{0,1\}^* : x \text{ does not have a substring } y \text{ such that } |N_0(y) - N_1(y)| = 3\}$ is regular.

   [10 points]

3. Does there exist an alphabet Σ and a language $L \subseteq \Sigma^*$ such that L is decidable but neither L nor $\overline{L}$ is context-free? Prove your answer. Remember, $\overline{L}$ is defined to be $\Sigma^* - L$.

   [20 points]

4. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}[x,y]$ the set of polynomials with integer coefficients in the variables x and y. Prove or disprove that the language $\{\langle p_1, \ldots, p_n\rangle : \text{each } p_i \in \mathbb{Z}[x,y] \text{ and } \exists\, u, v \in \mathbb{Z} \text{ such that } p_1(u,v) = \cdots = p_n(u,v) = 0\}$ is Turing-recognizable. Here $p(u,v)$ is the value of the polynomial p at the point (u,v).

   [10 points]

5. A language L over alphabet Σ is said to be *universally completable* if every string in Σ^* can be "completed" to a string in L; to be precise: $\forall x \in \Sigma^* \, (\exists y \in \Sigma^* \, (xy \in L))$. Thus, for example, the language $\{x \in \{0,1,2\}^* : N_0(x) = N_1(x) = N_2(x)\}$ is universally completable, because we can always "equalize" the number of 0's, 1's and 2's in any string by appending an appropriate suffix. On the other hand, the language $\{0^n1^n2^n : n \geq 0\}$ is not universally completable because, for instance, there is no string that starts with 01121 and is in this language.

   For each of the following computational problems, either describe an algorithm to solve the problem or prove, via undecidability, that no such algorithm exists.

   5.1. Given a regular expression R, determine whether $\mathcal{L}(R)$ is universally completable.

   [10 points]

   5.2. Given a context-free grammar G, determine whether $\mathcal{L}(G)$ is universally completable.

   [20 points]

6. Let $\textsc{HalfClique} = \{\langle G\rangle : G \text{ is an undirected } (2n)\text{-vertex graph and has a clique of size } n\}$. Prove that $\textsc{HalfClique}$ is NP-complete.

   [15 points]