Homework 7
Due Mar 4, 2004

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Just about every one of these problems can be solved by constructing an appropriate Turing Machine. When describing your TMs, use Sipser-style "on input $\langle$blah$\rangle$..." descriptions and keep them crisp; *do not write lengthy prose* or else you will lose credit.

As usual, please think carefully about how you are going to organise your answers *before* you begin writing. Make sure your answers are complete, clean, concise and rigorous.

1. Prove that every infinite Turing-recognizable language has an infinite decidable language as a subset. (The problem can be rephrased as follows: If L is an infinite recognizable language, then prove that there is an infinite language L' which is a subset of L and is decidable.) [15 points]

   Hint: Think of enumerator TMs and the results we proved in class about them.

2. All of the automata models we have studied in this course allow "useless" states, i.e., states which are never entered in any run. It would be nice to have an algorithm that could detect and prune such useless states in an automaton, but unfortunately this is not always possible! Define the languages

$$\begin{aligned}
\mathsf{U}_{\mathsf{DFA}} &= \{\langle M\rangle : M \text{ is a DFA and } M \text{ has at least one useless state}\}\,,\\
\mathsf{U}_{\mathsf{PDA}} &= \{\langle M\rangle : M \text{ is a PDA and } M \text{ has at least one useless state}\}\,,\\
\mathsf{U}_{\mathsf{TM}} &= \{\langle M\rangle : M \text{ is a TM and } M \text{ has at least one useless state}\}\,,
\end{aligned}$$

   Prove that

   2.1. $\mathsf{U}_{\mathsf{DFA}}$ is decidable. [5 points]

   2.2. $\mathsf{U}_{\mathsf{PDA}}$ is decidable. [10 points]

   2.3. U_{TM} is undecidable. [10 points]

3. For each of the following languages, classify the language into one of the following three categories: (a) not Turing-recognizable; (b) Turing-recognizable, but not decidable; (c) decidable.

   In each case, *prove* the correctness of your answer.

   3.1. $\{\langle M\rangle : M$ is a TM and M accepts at least two strings$\}$ [10 points]

   3.2. $\{\langle M\rangle : M$ is a TM and M accepts exactly two strings$\}$ [10 points]

   3.3. $\{\langle M\rangle : M$ is a TM and M halts when it runs on a tape that is initially empty$\}$
   (This is the same language as $\{\langle M\rangle : M$ is a TM and M halts on input $\varepsilon\}$.) [10 points]

   3.4. $\{\langle M_1, M_2\rangle : M_1$ and M_2 are TMs over the input alphabet $\{0,1\}$ and $\mathcal{L}(M_1) = \{0,1\}^* - \mathcal{L}(M_2)\}$
   [15 points]

4. Prove that the language $\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle : G_1$ and G_2 are context-free grammars and $\mathcal{L}(G_1) = \mathcal{L}(G_2)\}$ is co-Turing-recognizable (i.e., prove that the complement of $\mathsf{EQ}_{\mathsf{CFG}}$ is Turing-recognizable). [15 points]

Challenge Problems

CP7: Prove Rice's Theorem, i.e., do problem 5.22 from your textbook. Note that this theorem simplifies the solutions of some of the problems in this homework, but you are not allowed to use this theorem unless you have proved it yourself by solving this challenge problem.