

In class today, we discussed a proof that regular expressions can be converted into equivalent NFAs. The proof was by induction, and the heart of the inductive step consisted of three NFA constructions to show that if  $L_1$  and  $L_2$  are languages recognized by NFAs, then so are (1)  $L_1 \cup L_2$ , (2)  $L_1 L_2$ , and (3)  $L_1^*$ . These constructions are similar to those in your textbook. We gave *informal* proofs for why the constructions work, but not formal proofs. Here is how one should *formally* prove the correctness of the constructions.

**Theorem 1:** If  $L_1$  and  $L_2$  can be recognized by NFAs, then so can  $L_1 \cup L_2$ .

**Proof:** For  $1 \leq i \leq 2$ , let  $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$  be an NFA that recognizes  $L_i$ , and assume that  $Q_1 \cap Q_2 = \emptyset$ . Define a new state  $s \notin Q_1 \cup Q_2$  and an NFA  $M = (Q, \Sigma, \delta, s, F)$ , where

$$\begin{aligned} Q &= Q_1 \cup Q_2 \cup \{s\}, \\ F &= F_1 \cup F_2, \end{aligned}$$

and  $\delta$  is given by

$$\delta(q, a) = \begin{cases} \{q_1, q_2\}, & \text{for } q = s, a = \varepsilon, \\ \emptyset, & \text{for } q = s, a \in \Sigma, \\ \delta_1(q, a), & \text{for } q \in Q_1, a \in \Sigma_\varepsilon, \\ \delta_2(q, a), & \text{for } q \in Q_2, a \in \Sigma_\varepsilon. \end{cases}$$

We claim that  $\mathcal{L}(M) = L_1 \cup L_2$ . To prove this equality between sets, we must show two containments, as follows.

**(1)  $\mathcal{L}(M) \subseteq L_1 \cup L_2$ :** Suppose  $x \in \mathcal{L}(M)$ , i.e.,  $M$  accepts  $x$ . By definition, this means that we can write  $x = a_1 a_2 \cdots a_n$ , where each  $a_i \in \Sigma_\varepsilon$ , and there exists a sequence  $\langle r_0, r_1, \dots, r_n \rangle$  of states in  $Q$  such that the following three conditions hold.

- (1.1)  $r_0 = s$ ,
- (1.2)  $r_i \in \delta(r_{i-1}, a_i), \forall i$  with  $1 \leq i \leq n$ ,
- (1.3)  $r_n \in F$ .

By conditions (1.2) and (1.1), we have  $r_1 \in \delta(r_0, a_1) = \delta(s, a_1)$ . Therefore, by our construction of  $M$ , we must have  $a_1 = \varepsilon$  and  $r_1 \in \{q_1, q_2\}$ .

Consider the case when  $r_1 = q_1$ . Then  $r_1 \in Q_1$ . Repeatedly applying condition (1.2), we see that for all  $i$  with  $1 \leq i \leq n-1$ , we have  $r_i \in Q_1$  and thus, by our construction of  $M$ ,  $r_{i+1} \in \delta_1(r_i, a_{i+1}) \subseteq Q_1$ . Finally, by condition (1.3), we have  $r_n \in F = F_1 \cup F_2$ , but we also have  $r_n \in Q_1$ . Therefore,  $r_n \in F_1$ . We conclude that  $x = a_2 a_3 \cdots a_n$  and that the sequence  $\langle r_1, r_2, \dots, r_n \rangle$  satisfies

- (2.1)  $r_1 = q_1$ ,
- (2.2)  $r_i \in \delta_1(r_{i-1}, a_i), \forall i$  with  $2 \leq i \leq n-1$ ,
- (2.3)  $r_n \in F_1$ .

By definition, this means that  $M_1$  accepts  $x$ , so  $x \in L_1$ . Therefore  $x \in L_1 \cup L_2$ .

Consider the case when  $r_1 = q_2$ . By a similar argument, we conclude that  $x \in L_2$ . Therefore  $x \in L_1 \cup L_2$ .

Thus, in all cases,  $x \in \mathcal{L}(M)$  implies  $x \in L_1 \cup L_2$ . We conclude that  $\mathcal{L}(M) \subseteq L_1 \cup L_2$ .

**(2)  $L_1 \cup L_2 \subseteq \mathcal{L}(M)$ :** Think about how you would prove this. I leave it as an exercise.

**Theorem 2:** If  $L_1$  and  $L_2$  can be recognized by NFAs, so can  $L_1L_2$ .

**Proof:** For  $1 \leq i \leq 2$ , let  $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$  be an NFA that recognizes  $L_i$ , and assume that  $Q_1 \cap Q_2 = \emptyset$ . Define an NFA  $M = (Q, \Sigma, \delta, q_1, F_2)$ , where

$$Q = Q_1 \cup Q_2,$$

and  $\delta$  is given by

$$\delta(q, a) = \begin{cases} \delta_1(q, a), & \text{for } q \in Q_1, a \in \Sigma, \\ \delta_1(q, \varepsilon), & \text{for } q \in Q_1 - F_1, a = \varepsilon, \\ \delta_1(q, \varepsilon) \cup \{q_2\}, & \text{for } q \in F_1, a = \varepsilon, \\ \delta_2(q, a), & \text{for } q \in Q_2, a \in \Sigma_\varepsilon. \end{cases} \quad (*)$$

We claim that  $\mathcal{L}(M) = L_1L_2$ . To prove this equality between sets, we must show two containments, as follows.

(1)  $\mathcal{L}(M) \subseteq L_1L_2$ : Suppose  $x \in \mathcal{L}(M)$ , i.e.,  $M$  accepts  $x$ . By definition, this means that we can write  $x = a_1a_2 \cdots a_n$ , where each  $a_i \in \Sigma_\varepsilon$ , and there exists a sequence  $\langle r_0, r_1, \dots, r_n \rangle$  of states in  $Q$  such that the following three conditions hold.

$$(3.1) \quad r_0 = q_1,$$

$$(3.2) \quad r_i \in \delta(r_{i-1}, a_i), \forall i \text{ with } 1 \leq i \leq n,$$

$$(3.3) \quad r_n \in F_2.$$

Observe that  $r_0 \in Q_1$  and  $r_n \in Q_2$ . Therefore, there must exist an integer  $j$ , with  $0 \leq j \leq n-1$ , such that  $r_0, r_1, \dots, r_j \in Q_1$  and  $r_{j+1} \in Q_2$ . By construction, the only transitions from a state in  $Q_1$  to a state in  $Q_2$  are those given by (\*). It follows that  $r_j \in F_1$ ,  $r_{j+1} = q_2$  and  $a_{j+1} = \varepsilon$ . Again by construction, there are no transitions from a state in  $Q_2$  to a state in  $Q_1$ ; it follows that  $r_{j+1}, r_{j+2}, \dots, r_n \in Q_2$ .

We conclude that  $x = a_1a_2 \cdots a_j \varepsilon a_{j+2} \cdots a_n$  and that the sequences  $\langle r_0, r_1, \dots, r_j \rangle$  and  $\langle r_{j+1}, r_{j+2}, \dots, r_n \rangle$  satisfy the following sets of conditions.

$$(4.1) \quad r_0 = q_1,$$

$$(4.2) \quad r_i \in \delta_1(r_{i-1}, a_i), \forall i \text{ with } 1 \leq i \leq j,$$

$$(4.3) \quad r_j \in F_1.$$

$$(5.1) \quad r_{j+1} = q_2,$$

$$(5.2) \quad r_i \in \delta_2(r_{i-1}, a_i), \forall i \text{ with } j+2 \leq i \leq n,$$

$$(5.3) \quad r_n \in F_2.$$

By definition, conditions (4.1)–(4.3) mean that  $M_1$  accepts  $a_1a_2 \cdots a_j$  and conditions (5.1)–(5.3) mean that  $M_2$  accepts  $a_{j+2} \cdots a_n$ . Therefore, the former string belongs to  $L_1$  and the latter string belongs to  $L_2$ . Since the concatenation of these two strings is  $x$ , we get  $x \in L_1L_2$ . We conclude that  $\mathcal{L}(M) \subseteq L_1L_2$ .

(1)  $L_1L_2 \subseteq \mathcal{L}(M)$ : As before, I leave this half of the proof as an exercise. Think about it.

**Theorem 3:** If  $L$  can be recognized by an NFA, so can  $L^*$ .

**Proof :** If you have carefully read and understood the above material, and you remember the necessary construction from class, then writing out the necessary formal details should be an easy (but very worthwhile) exercise for you. I strongly encourage you to work this out. Of course, see me outside class if you have difficulty doing so.