Insolubility of the Problem of Homeomorphy*

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1. We consider, from the general problem of homeomorphy, the problem of finding an algorithm that determines whether two given polyhedra are homeomorphic. In this case, polyhedra are combinatorially given through their triangulation and we must understand the term "algorithm" in the precise sense what the it offers i.e., e.g., as a "classifying algorithm".

In addition to the general problem of the Homeomorphy, there are, of course, different subproblems which themselves refer to polyhedra or the those resulting classes. One may, for example, set up the problem of homeomorphy for polyhedra of degree no higher than n, a fixed natural number. One may, in exactly the same way, set up the problem of homeomorphy for the n-manifolds, if one could clearly decide what a "manifold" is.

Another natural restriction that can be made to the polyhedra to be matched is fixing one of them. In this case, the problem of the homeomorphy of a given polyhedron A consists of finding an algorithm which, for any polyhedron, determines whether it is homeomorphic to the polyhedron A.

One of these problems has been solved for a long time, i.e. the problem of homeomorphy for 2-manifolds or the problem of the homeomorphy of a given 2-manifold. However, we have found the following results:

Theorem 1 *For every natural number $n > 3$, one can create an n-manifold M^n, such that the problem of homeomorphy of manifolds to M^n is undecidable.*

Here, we use Poincaré's and Veblen's definition of a "manifold".

Corollary 1 The problem of homeomorphy of n-manifolds is undecidable for $n > 3$.

Corollary 2 The problem of homeomorphy for polyhedra of degree no higher than n is undecidable for $n > 3$.

Corollary 3 The general problem of homeomorphy is undecidable.

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2. Outline of the Proof for Theorem 1

Let K^4 be a 4-dimensional ball in the 4-dimensional spherical space S^4; S^3 its boundary; K^3, a 3-dimensional ball; I, the interval [-1,1] on the number line;

$$Z = K^3 \times I,$$

where $\times$ is the sign of topological multiplication and r is a natural number. We create a system of r pairwise-different differentiable representations $\phi_1, \ldots, \phi_r$, for the space Z in S^4, which has the the following characteristics:

$$\phi_i Z \cap \phi_j Z = \emptyset \quad (i, j = 1, \ldots, r; i \neq j)$$

$$\left.\begin{array}{l} K^4 \cap \phi_i Z \subset S^3 \\ S^3 \cap \phi_i Z = \phi_i(K^3 \times \{-1, 1\}) \end{array}\right\} (i = 1, \ldots, r),$$

where $\emptyset$ is the empty set.

We may form a polyhedron

$$L_r = K^4 \cup \textstyle\bigcup_{i=1}^r \phi_i Z,$$

in other words, we attach r handles to the 4-dimensional ball K^4

$$\phi_i Z (i = 1, \ldots, r).$$

In the space Z, linear sections form naturally. We agree to characterize the linear section through $[x, y]$ in Z with endpoints x and y. We also employ the same designation for the linear sections in the ball K^4 with endpoints x and y.

We define $2r$-letter alphabet:

$$\Gamma_r = \{\alpha_1^1, \ldots, \alpha_r^1, \alpha_1^{-1}, \ldots, \alpha_r^{-1}\}$$

Let P be a word over the alphabet Γ_r. We agreee to call every simple closed curve W, obtained as described below, an image of P. If P is empty, any arbitrary circumference that lies in the interior of K^4 may be taken as W.

Given P not empty and

$$P = \alpha_{i_1}^{\epsilon_1}, \ldots, \alpha_{i_s}^{\epsilon_s}, \tag{1}$$

where $i_1, \ldots, i_s$ are the numbers $1, \ldots, r$ and $\epsilon_j = \pm 1 (j = 1, \ldots, s)$. We choose in the interior of the ball K^3 the points $x_1, \ldots, x_s, y_1, \ldots, y_s$ so that the following conditions are fulfilled:

$$[(x_j, -\epsilon_j), (y_j, \epsilon_j)] \cap [(x_h, -\epsilon_h), (y_h, \epsilon_h)] = \emptyset \quad (j, h = 1, \ldots, s; j \neq h; i_j = i_h),$$

$$[\phi_{i_j}(y_j, \epsilon_j), \phi_{i_{j+1}}(x_{j+1}, -\epsilon_{j+1})] \cap [\phi_{i_h}(y_h, \epsilon_h), \phi_{i_{h+1}}(x_{h+1}, -\epsilon_{h+1})] = \emptyset \quad (j, h = 1, \ldots s; j \neq h),$$

where x_{s+1} is x_1 and y_{s+1} is y_1. This is always possible. Supposing that

$$W = \textstyle\bigcup_{j=1}^s (A_j \cup B_j)$$

where

$$\left.\begin{array}{rcl} A_j &=& \phi_{i_j}[(x_j,-\epsilon_j),(y_j,\epsilon_j)] \\ B_j &=& [\phi_{i_j}(y_j,\epsilon_j),\phi_{i_{j+1}}(x_{j+1},-\epsilon_{j+1})] \end{array}\right\} \quad (j=1,\ldots,s).$$

It is clear that the representation of every word over the alphabet Γ_r is a piece-wise linear simple closed curve in the interior of L_r.

The representation W of a word without blanks (1) may be built up just as shown. It characterizes c as the center of the ball K^3. The topological representation Ψ of the polyhedron $K^3 \times W$ in the interior of L_r may be built so that it has the following characteristics:

- T1. $\Psi(c,x) = x(x \in W)$.
- T2. Ψ is differentiable on every polyhedron $K^3 \times A_j$.
- T3. Ψ is differentiable on every polyhedron $K^3 \times B_j$.
- T4. $\Psi(K^3 \times A_j) \subset \phi_{i_j} Z$.
- T5. $\Psi(K^3 \times B_j) \subset K^4$.

If Ψ has these qualities, we will say over the interior of the quantities $\Psi(K^3 \times W)$, that it is a tunnel of the word P. For an empty word, the tunnels are defined analogously with one difference that Ψ must be the differentiable image of the polyhedron $K^3 \times W$ in the interior of the ball, which fullfills condition. It is not hard to see, that the tunnel of a word over the alphabet Γ_r is contained in the neighborhood of the image of that word. Let $P_1 * \cdots * P_m$ be a system of words over the alphabet Γ_r. We construct for each $i(1 \le i \le m)$ a tunnel T_i of the word P_i so that the following is satisfied:

$$\overline{T_i} \cap \overline{T_j} \quad = \quad \emptyset \quad (i,j=1,\ldots,m; i \neq j)$$

where the lines over the alphabetic characters characterizes the operation of closure in S^4. We construct a polyhedron

$$\begin{array}{rcll} J_0 &=& L_r \setminus \bigcup_{i=1}^m T_i \;, & \\ J_i &=& S^4 \setminus T_i & (i=1,\ldots,m), \\ H_{-1} &=& L_r \times \{-1\}, & \\ H_i &=& J_i \times \{i\} & (i=0,1,\ldots,m), \end{array}$$

Finally, we design the polyhedron M from the polyhedra $H_i(i=-1,0,\ldots,m)$ by gluing following points:

(1) the two points $(x,0)$ and $(x,-1)$ are glued, whenever x is on the boundary of L_r in S^4

(2) the two points $(x,0)$ and (x,i) are glued, whenever x lies on the boundary of the tunnels T_i in S^4, $(i=1,\ldots,m)$.

The described construction seems partly arbitrary. But the result of the construction, the polyhedron M, is unambiguous with respect to the homeomorphy of the defined word system $P_1 * \cdots * P_m$ and the number r. On the other hand, one can eliminate this arbitrariness easily. We characterize the resulting polyhedron with the symbol

$$\mathcal{M}(P_1 * \cdots * P_m * r).$$

One may easily see that this is always a 4-manifold. The construction of this polyhedron, as described, is a certain specification of the construction of a 4-manifold by Seifert and Trelfall with a given fundamental group.[1]. Using the methods presented in their book, we prove Lemma 1:

Lemma 1 There is always a word system $P_1 * \cdots * P_m$ over an alphabet Γ_r such that the fundamental group of the manifold $\mathcal{M}(P_1 * \cdots * P_m * r)$ is isomorphic to the group, defined by the system of interrelationships

$$P_i \quad \leftrightarrow \Lambda(i = 1, \ldots, m). \tag{2}$$

between the elements $\alpha_1^1, \ldots, \alpha_r^1$.

In this case, one considers the alphabetic characters $\alpha_1^{-1}, \ldots, \alpha_r^{-1}$ to be elements, opposite to the elements $\alpha_1^1, \ldots, \alpha_r^1$.[2] Further, one can prove the following lemmas about the homeomorphy of the manifold $\mathcal{M}(P_1 * \cdots * P_m * r)$:

Lemma 2 The manifolds $\mathcal{M}(P_1 * \cdots * P_m * r)$ and $\mathcal{M}(Q_1 * \cdots * Q_m * r)$ are homeomorphic if the word system $Q_1, \ldots, Q_m$ is the result of the substitution of the empty word instead of the insertion of the word $\alpha_i^\epsilon \alpha_i^{-\epsilon}(i = 1, \ldots, r; \epsilon = \pm 1)$.

Lemma 3 The manifolds $\mathcal{M}(P_1 * \cdots * P_m * r)$ and $\mathcal{M}(Q_1 * \cdots * Q_m * r)$ are homeomorphic if there exists a number i in $1, \ldots, m$ such that Q_i is the result of the cyclical interchanging of the alphabetic characters in the word P and that

$$Q_j \quad = \quad P_j \tag{3}$$

with $1 \leq j \leq m$ and $j \neq i$.

Lemma 4 The manifolds $\mathcal{M}(P_1 * \cdots * P_m * r)$ and $\mathcal{M}(Q_1 * \cdots * Q_m * r)$ are homeomorphic if there exists a number i in $1, \ldots, m$, such that Q_i is a group transformation from the word P_i and equality (3) holds with $1 \leq j \leq m$ and $j \neq i$.

We define the group transformation on the word P over the alphabet Γ_r as the transformation which results from reversing the alphabetic characters with subsequent replacement of every alphabetic character α_j^ϵ by the alphabetic character $\alpha_j^{-\epsilon}$.

Lemma 5 The manifolds $\mathcal{M}(P_1 * \cdots * P_m * r)$ and $\mathcal{M}(Q_1 * \cdots * Q_m * r)$ are homeomorphic if there exist numbers i and h in $1, \ldots, m$ with $i \neq h$, such that

$$Q_i \quad = \quad P_i P_h$$

and equality (3) holds with $1 \leq j \leq m$ and $j \neq i$.

Lemma 6 The manifolds $\mathcal{M}(*^k \alpha_1^1 * \cdots * \alpha_r^1 * r)$ and $\mathcal{M}(*^k 0)$ are homeomorphic, where k is a natural number.

[1]cf. 2, page 208

[2]For this group over the alphabet Γ_r, we can do the corresponding group calculations (cf [3], page 341) through a system of interrelationships, adding $\alpha_i^\epsilon \alpha_i^{-\epsilon} \leftrightarrow \Lambda(i = 1, \ldots, r); \epsilon = \pm 1$ to (2).

With help from lemma 2-6, we can easily prove the following:

Lemma 7 If the group with generating elements $\alpha_1^1, \ldots, \alpha_r^1$ that is determined by the system of relationships

$$R_i \quad \leftrightarrow \quad \Lambda(i = 1, \ldots, k) \tag{4}$$

over the alphabet Γ_r is the unit group, then manifold

$$\mathcal{M}(R_1 * \cdots * R_k *^{r+1} r)$$

is homeomorphic to the manifold $\mathcal{M}(*^k 0)$.

On the other hand, this follows from Lemma 1:

Lemma 8 If the group defined in Lemma 7 is not the unit group, then the manifold $\mathcal{M}(R_1 * \cdots * R_k *^{r+1} r)$ is not homeomorphic to the manifold $\mathcal{M}(*^k 0)$.

We may now fix natural numbers r and k. We will consider the group with the generating elements $\alpha_1^1, \ldots, \alpha_r^1$ that is defined by any system of k relations (4) over the alphabet Γ_r. We name these groups the (r, k)-groups. It follows from lemmas 7 and 8 that with the help of an algorithm which can determine whether a given manifold is homeomorphic to the manifold $\mathcal{M}(*^k 0)$, we may construct an algorithm which can determine whether an (r, k)-group is nontrivial. Meanwhile, it is immediate from the construction in the work of S. I. Adjan [1][3], that "numbers r and k can be given such that it is impossible to find an algorithm for recognizing the triviality of (r, k)-group". We take such a pair (r,k) and take the case that

$$M^4 \quad = \quad \mathcal{M}(*^k 0)$$

The problem of the homeomorphy of manifolds then proves itself to be unsolvable for the 4-dimensional M. It is not hard to see that for every natural number n greater than four, the problem of the homeomorphy of the n-manifold

$$M^n \quad = \quad M^4 \times S^{n-4}$$

is unsolvable, where S^h is the h-dimensional sphere. This ends the proof of Theorem 1.

3. We may look at homotopy equivalence problems in an analogous way. One gets the formulation from the formulation of the problem of homeomorphy by just replacing the word 'homeomorphy' with 'homotopy equivalent.' But this replacement is obviously possible in lemmas 7 and 8. This replacement yields the following results:

Theorem 2 *For each natural number n greater than 3, the problem of homotopy equivalence of manifolds to M^n is undecidable.*

Corollary 1 The problem of homotopy equivalence of n-manifolds is undecidable for $n > 3$.

Corollary 2 The problem of homotopy equivalence of polyhedra of degree no greater than n is undecidable for $n > 3$.

Corollary 3 The general problem of homotopy equivalence is undecidable.

[3] cf. also [6]