# Survey of Results on Minimal Triangulations

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**Abstract**

For any closed compact 2-manifold, there is a finite number of *minimal* (*irreducible*) triangulation. Thus, we may obtain any triangulation of a surface by splitting vertices of a minimal triangulation of the surface. The number of vertices in these minimal triangulations is linear in the genus of the underlying surface.

## 1 Introduction and Overview

For any closed surface, there are many possible triangulations with different number of triangles. These triangles, however, describe some surface and must reflect the inherent connectivity of the surface. As such, we'd like to know the minimum number of triangles (or equivalently vertices or edges) needed to describe a surface.

One approach to reaching the minimum number of triangles is to start with an arbitrary triangulation of a surface and repeatedly modify the triangulation in such a way that the result is smaller and still describes the surface. This is the approach taken by incremental mesh decimation algorithms such as [EN97]. We would hope that our algorithm results in a triangulation with the minimum number of triangles or at least in a triangulation that is *small enough.* In other words, we would like to employ a greedy strategy in optimizing the size of our triangulation and reach a local minimum that is close enough to the global minimum.

One operation for modifying a triangulation is contracting an edge. Intuitively, we choose an edge and contract it after checking that doing so does not change the topology. We call a triangulation reached by repeated *edge contractions* a *minimal* or *irreducible* triangulation. We would like the minimal triangulation to have size close to the minimum triangulation.

These interesting problems were formulated by Edelsbrunner as the open problem for Lecture #16 of a class on mesh generation [Ede98]:

Open Problem: Contraction-Minimal Manifold Let $\mathcal{K}$ be a triangulation of a 2-manifold $\mathbb{X}$. The map $\varphi_{ab} : |\mathcal{K}| \to |\mathcal{L}|$ contracts edge $ab$ in $\mathcal{K}$ preserves topological type iff $\mathrm{Lk}\, a \cap \mathrm{Lk}\, b = \mathrm{Lk}\, ab$. $\mathcal{K}$ is *minimal* if every edge contraction $\varphi_{ab}$, $ab \in \mathcal{K}$, changes topological type. $\mathcal{K}$ is *minimum* if $|\mathcal{K}'| = |\mathcal{K}| \Rightarrow \mathrm{card}\, \mathcal{K}' \geq \mathrm{card}\, \mathcal{K}$.

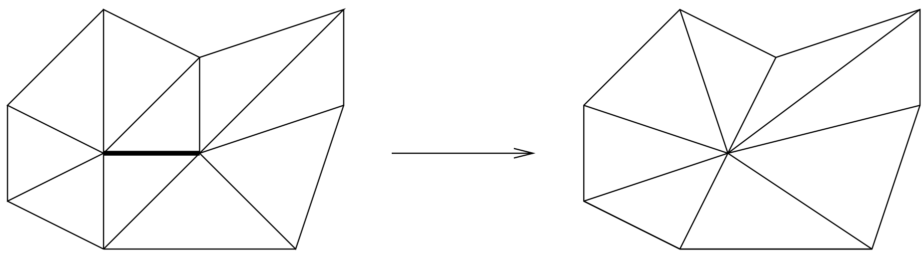
1. Does every minimal torus have 9 or fewer vertices?
2. Prove or disprove: there exists a constant $c$ such that $\mathrm{card}\, \mathcal{K} \leq c \cdot \mathrm{card}\, \mathcal{L}$ whenever $|\mathcal{K}| = |\mathcal{L}|$ and $\mathcal{K}$ is minimal.

In the remaining sections, we will look at results which give answers to both questions asked in the stated problem. I will first briefly describe the necessary concepts for minimal triangulations. I will then look at two interesting problems that arise: finding the number of minimal triangulations and bounding their size.

A note about notation: *minimum* triangulations are called *minimal* by Ringel who found an explicit formula of their sizes [JR80]. *Contraction-Minimal* triangulations are called *minimal* or *irreducible* triangulations by various authors.

## 2 Definitions

As the subject of minimal triangulations lies in the fields of topology and graph theory, there are various definitions for the same objects. I believe both views are necessary and will therefore mention both. A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex whose underlying space is homeomorphic to $\mathbb{X}$ [Ede98]. Equivalently, a triangulation of a closed surface is a simple graph embedded on the surface so that each face is a triangle and any two faces share at most one edge [NO95]. Let $\mathcal{K}$ be a triangulation and $ab \in \mathcal{K}$. $\mathcal{L}$ is $\mathcal{K}$ after contracting $ab$, where $\mathcal{L} = \mathcal{K} - \mathrm{St}\,\overline{ab} \cup c \cdot \mathrm{Lk}\,\overline{ab}$ [Ede98] (where St is the star, Lk is the link, bar is the closure, and $\cdot$ is the cone operation.) From a graph theoretical standpoint, contracting an edge $ab$ is removing $ab$, identifying the end vertices of $ab$, and replacing two pairs of multiple edges [NO95]. An edge contraction is shown below.



We would like to do an edge contraction only when there is no resulting change to the topology. Edelsbrunner [EN97] defines the *Link Condition* for a valid edge contraction: $\mathcal{L} \approx \mathcal{K} \Leftrightarrow \mathrm{Lk}\,a \cap \mathrm{Lk}\,b = \mathrm{Lk}\,ab$. A more intuitive version of this condition is the graph theoretical one. An edge $ab$ is *contractible* if it is contained in precisely two cycles of length 3, viewing the triangulation as a graph. If an edge is contained in more 3-cycles, then at least one 3-cycle does not bound a face and therefore we wouldn't want to contract the edge [NO95] (This argument works for spaces not homeomorphic to the 2-sphere.) Finally, we define a triangulation $\mathcal{K}$ of a surface to be *minimal* or *irreducible* if no edge is contractible.

## 3 Number of Minimal Triangulations

The minimal triangulations were initially studied for individual spaces. The results are as follows:

**(Steinitz 1934)** $K_4$ is the only minimal triangulation for $S^2$ [SR76].

**(Barnette 1982)** The projective plane has 2 minimal triangulations [Bar82].

**(Lavrenchenko 1987)** The torus has 21 minimal triangulations. All have 9 or 10 vertices [Lav87]. Several authors, including [BE89] state that B. Grünbaum and R. A. Duke proved this result earlier but didn't publish it. Also, K. Rusnak is given credit for independently proving it. Note that this result gives a negative answer to the first question of our stated open problem.

The general result for any surface can be derived from several fields.

**Graph Minors** In general, a graph $H$ is called a *minor* of a graph $G$ if $H$ can be obtained from $G$ via a *minor* operation, such as edge deletion or edge contraction. Wagner's conjecture states that in every infinite series of graphs, there exist a pair of graphs one of which is a minor of the other [NN93]. This conjecture was proven by Robertson and Seymour in 1996 [RS96]. We may prove immediately using this fact that every 2-manifold has finitely many minimal triangulations (use contradiction.) Graph minor theory, however, is quite nontrivial.

**$k$-minimal Triangulations** A triangulation is *$k$-minimal* (*$k$-irreducible*) if each edge is in a noncontractible cycle of length $k$ and is in no shorter noncontractible cycle. In particular, note that a minimal triangulation is a $k$-minimal triangulation with $k = 3$. Malnič and Nedela gave the first proof that the number of $k$-minimal triangulations for any surface is finite [MN95].

**Topology** Barnette and Edelson use topological concepts to prove that all orientable 2-manifolds have finitely many minimal triangulations [BE88] and later extend and simplify this result to all 2-manifolds [BE89]. I believe this is by far the most accessible of the results.

## 4 Size of Minimal Triangulations

We would like to find a tight bound on the size of minimal triangulations. Usually, we look at the number of vertices of the triangulation, but Euler's formula tells us that we could look at the number of edges or triangles equivalently. Note that bounding the size of minimal triangulations bounds their number.

Gao et. al. use $k$-minimal triangulation theory to give a proof that the number of vertices is at most $cr^4$ for a surface with Euler characteristic 2-$r$ (*Euler genus* $r$ and a constant $c$) [GRT91]. Later, they improve this bound to $cr^2$ and give explicit bound on the number of edges [GRS96]:

$$|E(T)| \quad \leq \quad 3k \cdot k!(6k)^k r^2.$$

For $k = 3$, we get $|E(T)| \leq 314928r^2$. The authors note that Euler's formula gives $|V(T)| = (1/3)|E(T)|+2-r$ and so for $k = 3$ we have the bound $|V(T)| \leq 104976r^2 + 2 - r$. But this bound is still quadratic in the genus.

Nakamoto and Ota give a linear bound for the number of vertices [NO95]:

$$|V(T)| \leq 171r - 72$$

They achieve such a bound using a theorem by Miller on edge amalgams [Mil87].

$G$ is a *vertex (edge) amalgam* of $H_1$ and $H_2$ if $G$ is obtained from disjoint graphs $H_1$ and $H_2$ by identifying a vertex (an edge) in $H_1$ with a vertex (an edge) in $H_2$ respectively. It's known that the oriented genus of a graph is additive over vertex amalgams but not edge amalgams. Miller defines the notion of a *generalized genus* of a graph as the connectivity of the surface for which the graph is embeddable. The connectivity of the surface equals the genus when the surface is unorientable and twice the genus if it is orientable. Miller then proves the following theorem:

**(Miller 1983)** The generalized genus of a graph is additive over vertex amalgams and over edge amalgams.

Nakamoto and Ota's result is a straightforward application of this theorem. They try to build a given triangulation $G$ via edge amalgams. If enough such amalgams are found, they use Miller's theorem to compute the genus. If not, they find a large independent set in $G$ to show the same result (this is where the constants 171 and 72 are derived.)

Nakamoto and Ota also build triangulations for orientable and nonorientable surfaces of arbitrary genus by using a minimal triangulation of a sphere on which they paste multiple copies of the largest minimal triangulation of a torus as handles in such a way that the resulting triangulation is minimal. These minimal triangulations have a linear number of vertices. Thus, the Nakamoto and Ota bound on the number of vertices is tight.

The factor of 171, however, seems rather large. An interesting question is whether a smaller factor can be found.

## 5 Acknowledgments

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