A Crash Course in Linear Programming

A general LP: \( \min c^T x : A x \geq b \)

- \( x \in \mathbb{R}^n \): variables
- \( A \in \mathbb{R}^{m \times n} \): constraint matrix, (usually \( m \geq n \))
- \( c \in \mathbb{R}^n \): obj. function

\( c^T x = c \cdot x = \langle c, x \rangle = \sum_{i=1}^n c_i x_i \).

Picture when \( n = 2 \):

\[ \begin{align*}
\min & \quad x_1 + 3x_2 \\
\text{s.t.} & \quad 3x_1 + 2x_2 \geq 6 \\
& \quad x_1 - x_2 \geq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*} \]

Linear Algebra Preliminaries

- Given \( A \), \( \{a_1, \ldots, a_m \} \subseteq \mathbb{R}^n \) are the \( m \)-rows
  \( \{A_1, \ldots, A_n \} \subseteq \mathbb{R}^m \) — \( m \)-cols.
• \( \text{Span} (v_1, \ldots, v_k) := \{ \sum_{i=1}^{k} \lambda_i v_i : \lambda_i \in \mathbb{R} \} \)

  This is an example of a Vector Space.

  • if \( \alpha \in \mathbb{R} \Rightarrow \alpha v \in V \)
  • \( v, \sigma \in V \Rightarrow v + \sigma \in V \)

• Lin. Ind: A set \( \{v_1, \ldots, v_k\} \) of vectors are lin. independent iff

  \[ \sum_{i=1}^{k} \lambda_i v_i = 0 \Leftrightarrow \lambda_i = 0 \text{ for } i = 1, \ldots, k \]

  eg: \( v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \)

  \( v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \)

  \( v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \)

• FACT: Any maximal collection of lin. ind. sets in a vector space has the same cardinality.

  The size of this “basis” is \( \dim(V) \).

  Try to prove this.

• Given a matrix \( A \), two important \( V \)-spaces

  1. Row-Space \( \equiv \text{Span} \{ a_1, \ldots, a_m \} \subseteq \mathbb{R}^n \)
  2. Col-Space \( \equiv \text{Span} \{ A_1, \ldots, A_n \} \subseteq \mathbb{R}^m \)

  \[ \text{row-rank} \equiv \dim(\mathbb{R}) ; \quad \text{col-rank} \equiv \dim(\mathbb{C}) \]

  \[ \max \# \text{ of lin-ind rows} \quad \max \# \text{ of lin-ind cols} \]
AMAZING FACT: \( \text{row-rank}(A) = \text{col-rank}(A) = \text{rank}(A) \)

Ways to think of \( Ax = \sum_{j=1}^{n} A_j x_j \) i.e. a linear comb of cols.

\[ Ax \in \mathbb{C}^m, \quad y^T A \in \mathbb{R} \]

\[ \forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^m. \]

**Proof:**

In \( \mathbb{R}^n \) (which is an inner-prod-space, i.e., it has an inner product defined on it), any \( V \)-space \( \leq \mathbb{R}^n \), has a “perpendicular” \( V \)-space \( V^\perp \) also.... \( V^\perp = \{ u : \langle u, v \rangle = 0 \quad \forall v \in V \} \)

**Fact:** \( \dim(V) + \dim(V^\perp) = n \)

\[ R^\perp = \{ x \in \mathbb{R}^n : (y^T A)x = 0, \quad \forall y \in \mathbb{R}^m \} \]

\[ = \{ x : Ax = 0 \} \]

Let \( \{ v_1, \ldots, v_d \} \) be a basis of \( R^\perp \)

\[ \leq \mathbb{R}^n \quad \text{note } d = n - \text{row-rank}(A) \]

This can be completed to a basis of \( \mathbb{R}^n \)
\[ B = \{ r_1, \ldots, r_d, s_{d+1}, \ldots, s_n \} \]

\[ \forall v \in \mathbb{R}^n, \quad v = \sum_{i=1}^{d} \lambda_i r_i + \sum_{i=d+1}^{n} \beta_i s_i \]

\[ \mathcal{C} = \{ Av \mid v \in \mathbb{R}^n \} \]

\[ = \{ A \sum_{i=1}^{d} \lambda_i r_i + \beta_{d+1}s_{d+1} + \cdots + \beta_n s_n \} \]

\[ = \{ A \cdot \sum_{i=1}^{d} \beta_i s_i \mid \beta_{d+1}, \ldots, \beta_n \} \]

\[ \text{dim } (\mathcal{C}) = n - d \]

\[ \begin{array}{ll}
\text{Col-rank} & \text{row-rank} \\
\end{array} \]

- A matrix \( A \) is said to have full row-rank if \( \text{row-rank } (A) = n \).

\[ \Rightarrow \mathbb{R}^n \equiv \{ 0 \} \]

Coming back to LP's:

- \( F = \{ x \mid Ax \geq b \} \) is called the feasible region.
- Given \( x \in F \), let \( B(x) \subseteq \{a_1, \ldots, a_m\} \) be the set of inequalities that hold with equality.

- Henceforth we assume \( A \) has full row-rank.

**BASIC FEASIBLE SOLN:**

Any \( x \in F \) is a basic feasible point if \( B(x) \) forms a basis of \( \mathbb{R}^n \).

Also called an **EXTREME POINT SOLN** or a **VERTEX solution**.

All bases of \( \{a_1, \ldots, a_m\} \) \( \leftrightarrow \) Basic Feasible Solutions.

\[
Bx = b_b \quad x = B^{-1}b_b
\]

**Thm:** Any LP has an Optimum Solution @ a basic feasible soln.

**Pf:** If \( x \) is an opt-soln & \( B(x) \) is full-row-rnk, then \( \exists v \in \mathbb{R}^n \) s.t. \( v^Ta_i = 0 \Rightarrow a_i \in B(x) \).
Consider \( x + \delta v \) \( \cdots \) finish the proof.

\[
\text{LOCAL-OPT} = \text{GLOBAL-OPT}
\]

- Fact: For any basis \( B \) of any \( \nu \)-space, \( \forall i \in B, \exists i' \in B \text{ at } B - i + i' \text{ is also a basis.} \)

- \( \text{cost} (B) := c^T x_B = c^T B^{-1} b_e \)

- Let \( B^* \) be the "local-opt" basis. i.e. \( \forall i \in B^*, \forall j \notin B^* \text{ if } \)
  \[ B_i = B^* - i + j \text{ is a basis}, \]
  then \( \text{cost} (B_i) \geq \text{cost} (B^*) \)

**Thm:** \( x_{B^*} \) is a Global Opt.

**Pf:** \( \tilde{x} := x_{B^*} = B^* b_{B^*} \)

\[ \forall i \in B^* \text{ \( x_i := x_{B_i} \)} \]

What is \( A(x_i - \tilde{x}) \) \( \mid \text{B}^* \) ?
it has the $i^{th}$ coor $> 0$
and rest all $0$

\[ A(x_i - \bar{x})\big|_{B^*} = \delta_i e_i \quad \text{for some} \quad \delta_i > 0 \]
\[ i = 1 \ldots n \]

\[ \Rightarrow \quad (x_i - \bar{x}) \text{'s span row-span}(A) = \mathbb{R}^n \]

\[ \Rightarrow \quad \text{Any } x \in F \text{ must satisfy } \]
\[ (x - \bar{x}) = \sum \alpha_i (x_i - \bar{x}) \]

\[ \text{with } \alpha_i \geq 0 \]

Why? Again multiply by $A$ and restrict all to $B^*$

\[ 0 \leq A(x - \bar{x})\big|_{B^*} = \sum \alpha_i \delta_i e_i \]
\[ \Rightarrow \quad \alpha_i \geq 0 \]

Picture:

Now we are done. If $\hat{x}$ is LOCAL OPT, then,

\[ c^T (x_i - \hat{x}) \geq 0 \]
if $x^*$ is global opt

$$0 \geq c^T(x^* - \bar{x}) = c^T\left(\sum_{i>0} d_i (x_i - \bar{x})\right)$$

\[
\geq 0
\]

\[
\Rightarrow \text{we have } c^T \text{ everywhere}
\]

\[
\Rightarrow c^T x^* = c^T x
\]

This is the "Simplex" method

... ok, not quite

... silly simplex method