Lecture 15

Thursday, May 11, 2017 4:45 PM

Cut Problems

① (s,t)- Min cut

Given graph G & (s,t), find minimum cost set of edges $F \subseteq E$ s.t. $G(V, E \setminus F)$ disconnects s from t.

② Multiway-Cut

Given $G = (V,E)$, costs on edges, and k-terminals $\{s_1, s_2, \ldots, s_k\}$, find min-cost colln of edges $F \subseteq E$ s.t. in $G' = (V, E \setminus F)$, every pair (s_i, s_j) is disconnected.

③ Multi-cut

Given $G = (V,E)$, costs on edges, and k source-sink pairs $\{(s_1,t_1), (s_2,t_2), \ldots, (s_k,t_k)\}$, find min-cost collection $F \subseteq E$ st in $G' = (V, E \setminus F)$, s_i is disconnected from t_i (but s_i could be conn. to t_j)

Min-cut

LP-relxn: $\min \sum c_e x_e$

$$\forall p \in P_{st} : \sum_{e \in p} x_e \geq 1$$

$$x_e \in [0,1]$$

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- Given x, one can define a metric d

- The above LP has a sep. oracle and can be solved.
- Given x one can define a metric d_x

$$d_x(u,v) := \text{shortest path dist from } u \rightsquigarrow v \text{ with } x\text{-costs on edges.}$$

- In fact, this motivates the following COMPACT LP:

Var $d(u,v)$ $\forall$ pairs u,v

$$\boxed{\begin{aligned} &\min \sum c_e\, d(e) \\ &- d(s,t) \geq 1 \\ &- d(u,v) \leq d(u,v) + d(v,w) \quad \forall u,v,w \end{aligned}}$$

ALGORITHM:

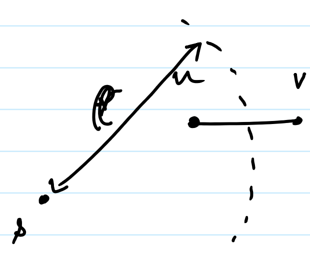
- choose $\rho \in_R [0,1]$
- $S := \{u : d(s,u) \leq \rho\}$
- Return $F = \partial S := \{(u,v) : |\{u,v\} \cap S| = 1\}$

ANALYSIS

$$X_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{o/w} \end{cases}$$

$$\mathbb{E}\,ALG = \sum_{e \in E} c_e \cdot \mathbb{E}[X_e]$$

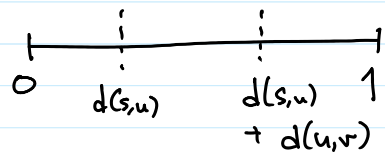
Fix $e = (u,v)$.
WLOG, assume $d(s,u) \leq d(s,v)$



$e \in \partial S \Rightarrow d(s,u) \leq \rho$ & $d(s,v) > \rho$

But $d(s,v) \leq d(s,u) + d(u,v)$

$\therefore e \in \partial S \Rightarrow d(s,u) \leq \rho \leq d(s,u) + d(u,v)$



$$\Pr_{\rho \in [0,1]}\left[e \in \partial S\right] \leq \Pr_{\rho \in [0,1]}\left[\rho \in \left[d(s,u),\, d(s,u) + d(u,v)\right]\right]$$

Since ρ is uniform in $[0,1]$, the prob. it is in any <u>fixed</u> interval of length ℓ is $\leq \ell$.

$$\therefore \Pr\left[e \in \partial S\right] \leq x_e$$

$$\therefore \mathbb{E}\,ALG \leq \sum c_e x_e = LP$$

∎

<u>Multiway Cut</u>

<u>ALGORITHM</u>

- Select $\rho \in_R (0, \frac{1}{2})$
- $S_i := \{u \mid d(s_i, u) \leq \rho\}$
- Return $F = \bigcup_{i=1}^{k} \partial S_i$

<u>ANALYSIS</u>

As before, $X_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{o/w} \end{cases}$

& we want to upper bound $\mathbb{P}[X_e = 1]$

Fix $(u,v) = e$.

Obs :- For any u, there can be at most one $i = 1..k$ st $d(s_i, u) < 1/2$

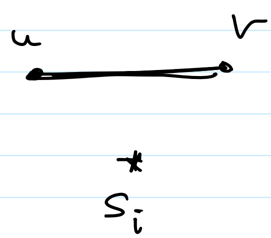
$\because$ if $d(s_i, u) < \frac{1}{2}$ & $d(s_j, u) < \frac{1}{2}$

$$\Rightarrow d(s_i, s_j) < 1 \quad \times$$

Let s_u (if any) be the terminal with $d(s_u, u) < \frac{1}{2}$ &

similarly let s_v be the terminal (if any) s.t. $d(s_v, v) < \frac{1}{2}$.

Case 1 : $S_u = S_v = S_i$



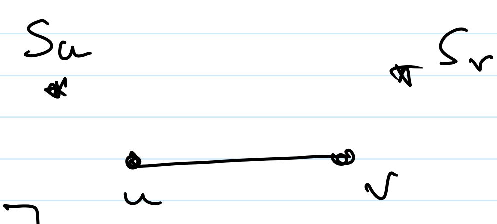
Then $\Pr[e \in F]$

$= \Pr[e \in \partial S_i]$

$\leq \Pr\left[d(s_i, u) < \rho < d(s_i, u) + d(u, v)\right]$ once again WLOG assume $d(s_i, u) \leq d(s_i, v)$

$\leq 2 \cdot d(u, v)$ since $\rho \in_R \left(0, \frac{1}{2}\right)$

Case 2 :- $S_u \neq S_v$



$\Pr[e \in F]$

$= \Pr[e \in \partial S_u \text{ OR } e \in \partial S_v]$

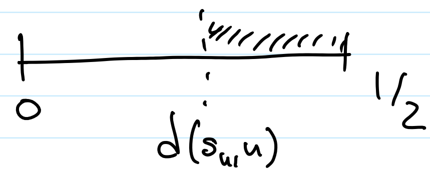
$\leq \Pr[e \in \partial S_u] + \Pr[e \in \partial S_v]$

When is $e \in \partial S_u$?

Note: $d(s_u, v) > \frac{1}{2}$ $\because s_u \neq s_v$

$\therefore$ v is outside S_u for sure.

$$\therefore \quad e \in \partial S_u \iff d(s_u, u) < \rho$$



$$\therefore \Pr[e \in \partial S_u] = \frac{1/2 - d(s_u,u)}{1/2}$$

$$= 1 - 2d(s_u, u)$$

Similarly,

$$\Pr[e \in \partial S_v] = 1 - 2d(s_v, v)$$

$$\therefore \Pr[e \in F] \leq 2\{1 - d(s_u,u) - d(s_v,v)\}$$

$$\leq 2d(u,v)$$

$$\because 1 \leq d(s_u, s_v) \leq d(s_u,u) + d(u,v) + d(s_v,v)$$

$\blacksquare$

Multicut

$$\min \sum_{e \in E} c(e)\, d(e)$$

$$\forall i = 1..k, \quad d(s_i, t_i) \geq 1$$

$$d \text{ sat. } \Delta\text{-ineq.}$$

ALGO:

1. Pick σ a permutation of $\{1, 2, .., k\}$ uniformly at random.
2. Pick $\rho \in_R (0, \frac{1}{2})$
3. For $i = 1, \dots, k$:

$$S_{\sigma(i)} := \{u \mid d(s_{\sigma(i)}, u) \leq \rho\} \setminus \bigcup_{j<i} S_{\sigma(j)}$$

Note: $s_{\sigma(i)}$ maynot be in $S_{\sigma(i)}$ but in a set $S_{\sigma(j)}$ for $j<i$.

4. Return $F = \bigcup_{i=1}^{k} \partial S_{\sigma(i)}$

ANALYSIS

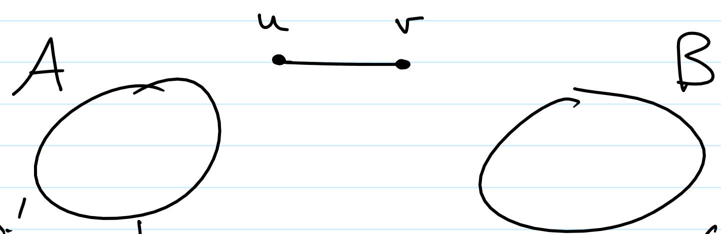
Claim: F is a valid multicut.

Pf: s_i & t_i can't be in $S_{\sigma(j)}$
$\because$ $d(s_i, s_{\sigma(j)}) + d(t_i, s_{\sigma(j)})$ would then be < 1.

Cost Analysis:

Fix an edge $e = (u,v)$. $X_e = 1$ if $(u,v) \in F$

We are interested in upper bounding $\mathbb{P}[X_e = 1]$.



S_i's s.t. $d(s_i, u) \leq d(s_i, v)$

$d(s_i, v) \leq d(s_i, u)$

$$\Pr[(u,v) \in F] = \Pr[\exists j : (u,v) \in \partial S_j \ \& \ (u,v) \notin \partial S_{j'} \text{ for all } j' \lneq j]$$

$$\leq \Pr[\exists j \in A : \text{ — '' — }] + \Pr[\exists j \in B : \text{ — '' — }]$$

Let's consider the first term.

$$\Pr[\exists j \in A : (u,v) \in \partial S_j \ \& \ (u,v) \notin \partial S_{j'} \text{ for all } j' \lneq j]$$

$$\leq \sum_{j \in A} \Pr[d(s_j, u) \leq \rho < d(s_j, v) \ \underline{\textbf{AND}} \ u \notin S_{j'} \text{ for some } j' : j' \lneq j] \quad \ldots \circledast$$

Let $A = \{1, 2, \dots, r\}$ s.t.

$$d(s_1, u) \le d(s_2, u) \le \dots \le d(s_r, u)$$

if $d(s_j, u) \le \rho \Rightarrow d(s_1, u) \le \dots \le d(s_j, u) \le \rho$

$\therefore$ if the above event were to occur, $j <_\sigma \{1, 2, \dots, j-1\}$ for if $j' \le j-1$ were s.t. $j' <_\sigma j$ then $u \in S_{j'}$

$$\therefore \ (*) \le \sum_{j \in A} \Pr\Big[d(s_j, u) \le \rho < d(s_j, v) \ \ \underline{\text{AND}} \ \ j <_\sigma \{1, 2, \dots, j-1\}\Big]$$

ARE NOW INDEPENDENT!

$$= \sum_{j \in A} \Pr_\rho\Big[d(s_j, u) \le \rho < d(s_j, v)\Big] \cdot \Pr_\sigma\Big[j <_\sigma \{1, 2, \dots, j-1\}\Big]$$

$$\le 2\, d(u, v) \sum_{j \in A} \frac{1}{j}$$

$$\le 2 \cdot d(u, v) \cdot H_r \le 2 d(u, v) \ln k$$

$\therefore$ Together we get, $\forall e = (u, v)$

$$\mathbb{P}[e \in F] = O(\ln k) \cdot d(u,v)$$

implying

$$\mathbb{E}\,ALG = O(\ln k) \cdot LP$$

$\blacksquare$