Lecture 17

Thursday, May 18, 2017 4:03 PM

Semidefinite Programming Relaxations

Example : MAX-CUT

Input : • $G = (V,E)$; wts on edges

O/p : • $S \subseteq V$; maximize $w(\partial S)$.

A "Quadratic" formulation :

Var : $x_i \in \{+1, -1\}$

$$OPT = \text{Maximize } \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - x_i x_j)$$

$$x_i^2 = 1 \qquad \forall i \in V.$$

· Solving QPs is NP-hard.

$$= \text{Maximize } \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - X_{ij})$$

- $X_{ii} = 1 \qquad \forall i = 1 \ldots n$
- $X = xx^T$ → "outer" product
- $x \in \mathbb{R}^n$

hard constraint

rank 1 matrix with $M_{ij} = x_i x_j$

Not "Convex" : X is xx^T $\quad Y = yy^T$

in general, $\alpha X + \beta Y \neq vv^T$ for some v

Relaxation

$$\text{Maximize } \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - X_{ij})$$

$$X_{ii} = 1 \qquad \forall i = 1 \ldots n$$

[" X is a sum of outer products "]

$$|||$$

$$X \succcurlyeq 0$$

or

X is a positive semidefinite matrix.

PSD matrices:

Defn: X is PSD if $X = \sum_{i=1}^{k} u_i u_i^T$

for some $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$

Properties: (0) X is PSD $\Rightarrow X = X^T$ (def^n)

(1) X is PSD $\Rightarrow v^T X v \geq 0 \quad \forall v \in \mathbb{R}^n$

$$v^T X v = v^T \left(\sum_{i=1}^{k} u_i u_i^T \right) v$$

$$= \sum_{i=1}^{k} (u_i^T v)^2 \geq 0$$

(2) If X is symmetric & $v^T X v \geq 0$, $\forall v$

then X has non-negative eigenvalues.

X is symm $\Rightarrow$ all eigenvalues/vecs of X are real

$$X u_i = \lambda_i u_i \qquad \forall i = 1..n$$

with $\|u_i\| = 1$

$$\therefore \quad u_i^T X u_i = \lambda_i \|u_i\|^2 = \lambda_i \Rightarrow \lambda_i \geq 0$$

(3) X is symm & has non-negative eigenvalues

$\Rightarrow X$ is PSD (sum of outer prods)

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal eigenbasis.

then $X = \sum_{i=1}^{n} \lambda_i u_i u_i^T$

Can assume the outer products to be of

orthogonal vecs.

④ X is PSD $\Rightarrow$ $X = V^T V$ for some matrix V [Cholesky Decomp.]

$$X = \sum_{i=1}^{n} u_i u_i^T \Rightarrow X_{ij} = \sum_{t=1}^{n} u_t(i)\, u_t(j)$$

Define $\underbrace{v_i(t)}_{t^{th} \text{ coor of } v_i} := \underbrace{u_t(i)}_{i^{th} \text{ coor of } u_t}$

$$\langle v_i, v_j \rangle = \sum_{t=1}^{n} v_i(t)\, v_j(t)$$
$$= \sum_{t=1}^{n} u_t(i)\, u_t(j) = X_{ij} \qquad \blacksquare$$

Facts: $X \succeq 0$, $Y \succeq 0$

① $\alpha X \succeq 0$ for $\alpha \geq 0$

② $X + Y \succeq 0$

They form a "convex cone"

THEREFORE WE CAN OPTIMIZE OVER IT.

$$SDP_{max\text{-}cut} = \text{Max} \quad \frac{1}{2} \sum_{(i,j) \in E} w_{ij} (1 - X_{ij})$$
$$X_{ii} = 1 \quad \forall i$$
$$X \succeq 0$$

can be calculated in poly-time.

ROUNDING

- Given a graph G, we can solve the sdp above to get $X \succeq 0$
- Cholesky Decomposition: $X = V^T V$

  <u>ie</u> $\exists\ v_1, v_2, \ldots, v_n$

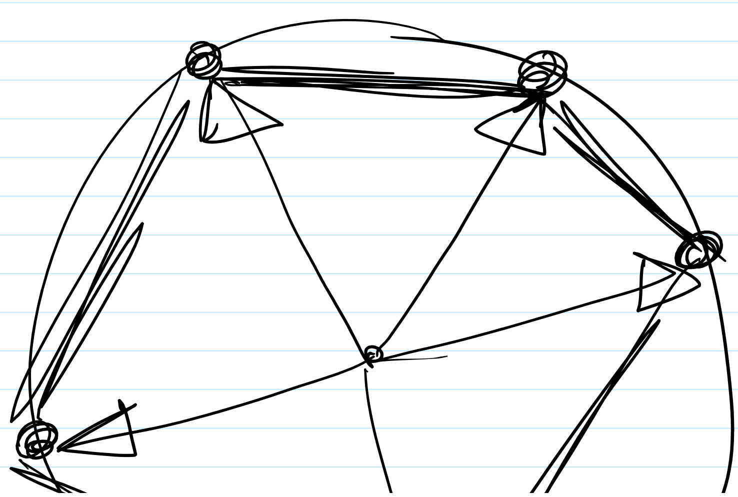
  (a vector for every vertex)

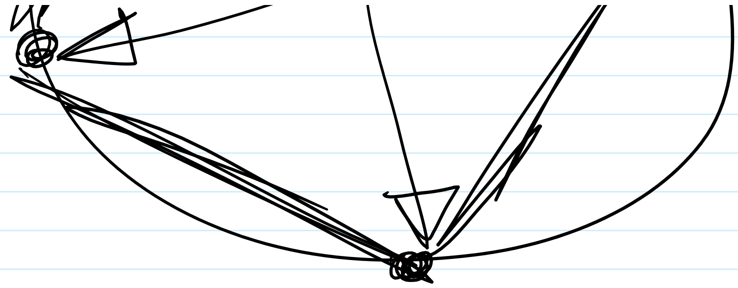
  s.t. $X_{ij} = \langle v_i, v_j \rangle$

$\therefore$ sdp $= \max \frac{1}{2} \sum w_{ij} \left(1 - \langle v_i, v_j \rangle\right)$

$$\|v_i\|^2 = 1, \quad \forall i$$

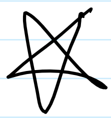
would like to put v_i & v_j in "antipodal" pts.

→ 



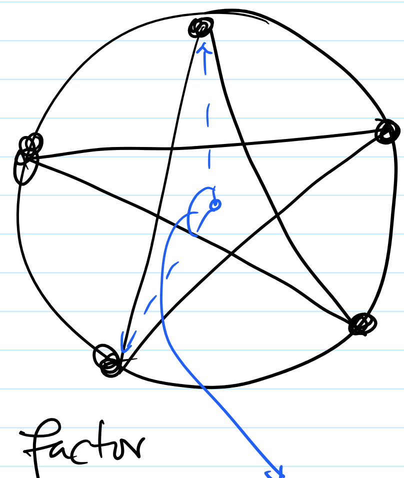
Example :- For the 5-cycle C_5, $W \equiv 1$

we could put the 5 points in a

 form

$$sdp \geq \frac{1}{2} \cdot 5 \cdot \left(1 - \cos\left(\frac{4\pi}{5}\right)\right)$$

$$\approx 4.523$$



∴ Can't use this for better than $\frac{4}{4.523} \approx 0.88$ factor

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Goemans-Williamson Rounding

→ Solve SDP & Cholesky Decomposition to obtain $\{v_1, v_2, \ldots, v_n\}$

$$\|v_i\| = 1$$

→ Sample q : a random unit vector

→ Sample g : a random unit vector in n-dimensions

(More on this in a bit)

→ $S := \{ i \mid \langle v_i, g \rangle \geq 0 \}$

Return S

Analysis :-

Again, by Linearity of expectation, it suffices to lower bound

$$\Pr\left[(i,j) \in \partial S \right]$$

for every $(i,j) \in E$

- Fix i, j & focus on the plane spanned by v_i, v_j
- Let g_{ij} be the projection of

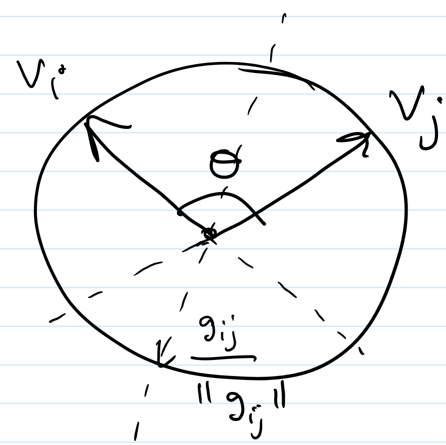
g on this plane.

OBS :- Since g is rotationally invariant, g_{ij} is also rotationally invariant.

$\therefore \Pr[(i,j) \in \partial S]$

$= \Pr[\langle v_i, g_{ij}\rangle$ isn't the same sign as $\langle v_j, g_{ij}\rangle]$

$= \dfrac{\theta}{\pi}$



where θ is the angle betn v_i & v_j.

$$\therefore \Pr[(i,j) \in \partial S] = \frac{\cos^{-1}(\langle v_i, v_j\rangle)}{\pi}$$

LP gets $\frac{1}{2}(1 - \langle v_i, v_j\rangle)$ from this edge.

$\therefore$ If we define

$$\alpha_{GW} := \frac{2}{\pi} \inf_{\substack{v,w \\ \|v\|=\|w\|=1}} \frac{\cos^{-1}(\langle v,w\rangle)}{1-\langle v,w\rangle}$$

then

$$\mathbb{E}[ALG] \geq \alpha_{GW} \cdot SDP$$

$\approx 0.878\ldots$