Semidefinite Programming Relaxations

Example: MAX-CUT

Input: \( G = (V, E) \); \( G \) has an edge
Output: \( S \subseteq V \); maximize \( \omega(S) \).

A "Quadratic" formulation:

\[
\text{Var: } x_i \in \{\pm 1\}
\]

\[
\text{OPT = Maximize } \frac{1}{2} \sum_{(i,j) \in E} w_{ij} (1 - x_i x_j)
\]

\[x_i^2 = 1 \quad \forall i \in V.\]

Solving QPs is \( \text{NP-hard} \).

\[
= \text{Maximize } \frac{1}{2} \sum_{(i,j) \in E} w_{ij} (1 - x_{ij})
\]

\[x_{ii} = 1 \quad \forall i = 1 \ldots n\]

\[x = xx^T \quad \text{"outer" product}\]

\[x \in \mathbb{R}^n \quad \text{rank 1 matrix} \quad \text{with } \omega_{ij} = x_i x_j\]

Not "Convex": \( x \) is \( xx^T \quad y = yy^T\)

in general, \( \alpha x + \beta y \neq \nu \nu^T \) for some \( \nu\)

Relaxation

\[
\text{Maximize } \frac{1}{2} \sum_{(i,j) \in E} w_{ij} (1 - x_{ij})
\]

\[x_{ii} = 1 \quad \forall i = 1 \ldots n.\]
[“X is a sum of outer products”]

\[ \| X \| \geq 0 \]

\[ X \text{ is a positive semidefinite matrix.} \]

**PSD matrices:**

**Defn:** \( X \) is PSD if \( X = \sum_{i=1}^{k} u_i u_i^T \)

for some \( u_1, u_2, \ldots, u_k \in \mathbb{R}^n \)

**Properties:**

1. \( X \) is PSD \( \Rightarrow \) \( X = X^T \) (def)

2. \( X \) is PSD \( \Rightarrow \) \( \nu^T X \nu \geq 0 \) \( \forall \nu \in \mathbb{R}^n \)

\[
\nu^T X \nu = \nu^T \left( \sum_{i=1}^{k} u_i u_i^T \right) \nu = \sum_{i=1}^{k} (u_i^T \nu)^2 \geq 0
\]

3. If \( X \) is symmetric and \( \nu^T X \nu \geq 0 \), \( \forall \nu \)

then \( X \) has non-negative eigenvalues.

\( X \) is symm \( \Rightarrow \) all eigenvalues of \( X \) are real

\[
X u_i = \lambda_i u_i \quad \forall i:1...n
\]

with \( \| u_i \| = 1 \)

\[
\vdots \quad u_i^T X u_i = \lambda_i \| u_i \|^2 = \lambda_i \Rightarrow \lambda_i \geq 0
\]

4. \( X \) is symm and has non-negative eigenvalues

\( \Rightarrow \) \( X \) is PSD (sum of outer products)

Let \( \{ u_1, u_2, \ldots, u_n \} \) be an orthonormal eigenbasis.

then \[
X = \sum_{i=1}^{n} \lambda_i u_i u_i^T
\]

Can assume, the outer products to be.
1. $X$ is PSD $\Rightarrow$ $X = V^T V$ for some matrix $V$ [Cholesky Decomposition]

$$X = \sum_{i=1}^{n} u_i u_i^T \Rightarrow X_{ij} = \sum_{t=1}^{n} u_t(i) u_t(j)$$

Define $v_i(t) := u_t(i)$, the $t^{th}$ coor of $v_i$, $u_t(i)$, the $i^{th}$ coor of $u_t$

$$\langle v_i, v_j \rangle = \sum_{t=1}^{n} v_i(t) v_j(t) = \sum_{t=1}^{n} u_t(i) u_t(j) = X_{ij}$$

Facts:

$X \succeq 0, Y \succeq 0$

1. $\alpha X \succeq 0$ for $\alpha \geq 0$

2. $X + Y \succeq 0$ "Convex cone"

Therefore we can optimize over it.

$$\text{SDP}_{\text{max-cut}} \quad = \quad \text{Max} \quad \frac{1}{2} \sum_{(i,j) \in E} w_{ij} (1 - X_{ij})$$

$$X_{ii} = 1 \quad \forall i$$

$X \succ 0$
can be calculated in poly-time.

**ROUNDING**

- Given a graph $G$, we can solve the SDP above to get $X \succeq 0$.
- Cholesky Decomposition: $X = V^T V$

  $i.e.$ $\exists v_1, v_2, \ldots, v_n$

  (a vector for every vertex)

  st. $X_{ij} = \langle v_i, v_j \rangle$

  $\Rightarrow$ SDP $= \max \frac{1}{2} \sum_{ij} W_{ij} \left( 1 - \langle v_i, v_j \rangle \right)$

  $\|v_i\|^2 = 1$, for $i$

would like to put $v_i$ & $v_j$ in "antipodal" pts.
Example: For the 5-cycle $C_5$, we could put the 5 points in a star form.

\[
\text{SDP} \geq \frac{1}{2} \cdot 5 \cdot \left(1 - \cos \left(\frac{4\pi}{5}\right)\right) \\
\approx 4.523
\]

* Can't use this for better than \( \frac{4}{4.523} \approx 6.88 \) factor

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**Goemans-Williamson Rounding**

→ Solve SDP & Cholesky Decomposition to obtain $\frac{1}{2}v_1, v_2, \ldots, v_n$ where $||v_i|| = 1$

→ Sample $q$ : a random unit vector.
Sample \( g \): a random unit vector in \( n \)-dimensions

(More on this in a bit)

\[ S := \exists i \mid \langle \nu_i, g \rangle \geq 0 \]

Return \( S \)

Analysis:

Again, by linearity of expectation, it suffices to lower bound

\[ \Pr \left[ (i, j) \in S \right] \]

for every \( (i, j) \in E \)

* Fix \( i, j \) & focus on the plane spanned by \( \nu_i, \nu_j \)

* Let \( g_{ij} \) be the projection of
$g$ on this plane.

**OBS**: Since $g$ is rotationally invariant, $g_{ij}$ is also rotationally invariant.

$$\Pr [(i,j) \in ES]$$

$$= \Pr [\langle v_i, g_{ij} \rangle \text{ isn't the same sign as } \langle v_j, g_{ij} \rangle]$$

$$= \frac{\theta}{\pi}$$

where $\theta$ is the angle between $v_i$ and $v_j$.

$$\Pr [(i,j) \in ES] = \frac{\cos^{-1}(\langle v_i, v_j \rangle)}{\pi}$$

LP gets $\frac{1}{2} (1 - \langle v_i, v_j \rangle)$ from this edge.
If we define

\[ \chi_{GW} = \frac{2}{\pi} \inf_{v, w \in V} \frac{\cos^{-1}(\langle v, w \rangle)}{1 - \langle v, w \rangle} \]

then

\[ \mathbb{E}[\text{ALG}] \geq \chi_{GW} \cdot \text{SDP} \]

\[ \approx 0.878 \ldots \]