Lecture 7

Saturday, April 15, 2017 7:36 PM

Perfect-Matching Polytope in Bip. Graphs

$$
\left\{ x_{uv} \ (\in [0,1]) : (u,v) \in G \right.
$$

$$
\forall u \in A : \sum_{v : (u,v) \in E} x_{uv} = 1 \qquad \text{(PM)}
$$

$$
\left. \forall v \in B : \sum_{u : (u,v) \in E} x_{uv} = 1 \right\}
$$

$\boxed{\textbf{Thm:} \text{ Polytope PM is integral.}}$

Pf: Let x be any basic feasible solution. We want to prove $x_{uv} \in \{0,1\}$.

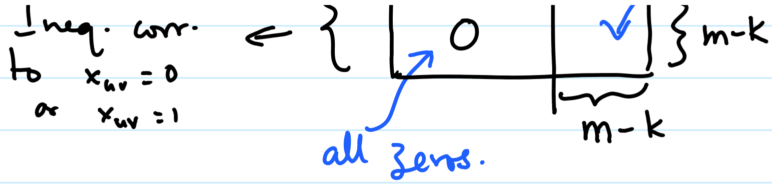
Let B be the set of inequalities that x satisfies with equality. We know,

$$
\text{Rank}(B) = m \leftarrow \#\text{ of edges.}
$$

Let $F := \{uv \mid 0 < x_{uv} < 1\}$.
Suppose $|F| = k > 0$.
Let $I = E \setminus F$ be the remaining integral vars.

Finally, let us re-order columns of B to obtain the following:

$$
B = \begin{array}{r|c|c|l}
 & F & I & \\ \hline
\text{Corresponds to } \sum_u x_{uv} = 1 & B_F & & \\ \hline
\text{Ineq. corr. to } x_{uv} = 0 & 0 & \text{Identity matrix} & \Big\} m-k \\ \hline
\end{array}
$$



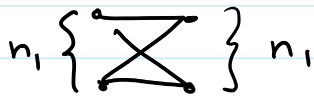
<u>Observe</u> :- $\text{rank}(B) = \text{rank}(B_F) + m-k$

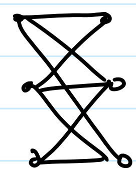
Convince yourself of this.

$\therefore \ \text{rank}(B_F) = k$ "The rank of the matrix corr to 'interesting' constraints & frac vars = # of frac vars"

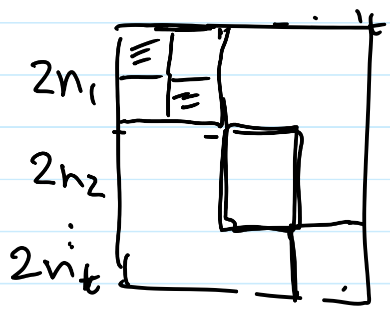
Let's focus on B_F

- Many conn. components
- Correspond to "square" submatrices of B_F





Why should each conn comp. have equal # on both sides?



- $\text{rank}(B_F|_i) = 2n_i - 1$ .... rows of the two n_i halves sum to the same.

  $\therefore \ \text{rank}(B_F) = 2\sum_{i=1}^{t} n_i - t$

- However, $|F| \geq 2\sum n_i \Rightarrow \text{rank}(B_F) \leq k-t$

  $\therefore \ t = 0 \Rightarrow$ no conn. components

  $\Rightarrow k = 0$ ∎

<u>Makespan Minimization on Unrelated Machines</u>

Makespan Minimization on Unrelated Machines

Input: m - machines
n - jobs
p_{ij} - proc time of job j on machine i

Output:- Assignment of jobs to machines

Objective:- To minimize makespan $\equiv \max_i \sum_{j \mapsto i} p_{ij}$

Special Case: $p_{ij} = \{p_j, \infty\}$

Each job has intrinsic proc time p_j but is restricted only on some machines.
(**Restricted Assignment Makespan Minimization**)

LP-relaxation

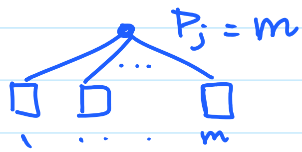
Min T

$\forall i = 1..m, \quad T - \sum_j p_{ij} x_{ij} \geq 0$

$\forall j = 1, \dots, n, \quad \sum_i x_{ij} = 1$

As stated above, this LP is bad.

Example:



- $x_{ij} = 1/m \quad \forall i$
- $T = 1$
- $OPT = m$

"Guess" the optimum $T \geq p_{max}$.

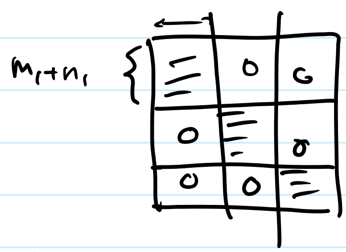
Rather, use only the fact the following polytope is feasible (given T)

$$\left.\begin{aligned} \forall i = 1 \dots m, \quad & \sum_{j=1}^{n} p_{ij} x_{ij} \le T \\ \forall j = 1, \dots, n, \quad & \sum_{i=1}^{m} x_{ij} = 1 \\ & 0 \le x_{ij} \le 1 \end{aligned}\right\} LP(T)$$

- x be a bfs of $LP(T)$ ...

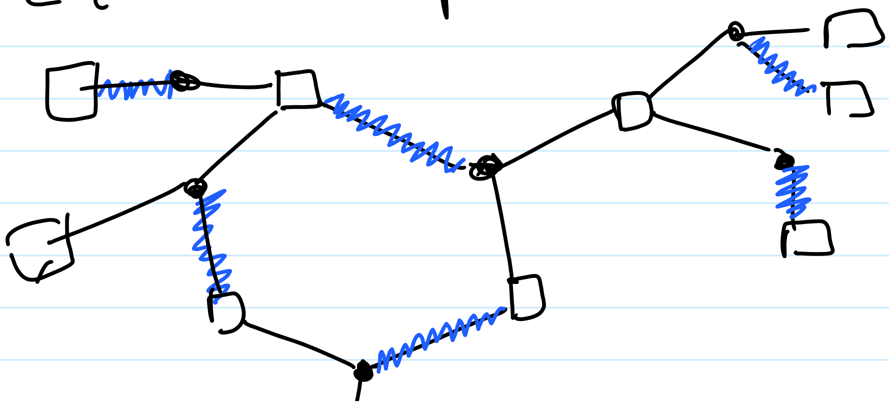
> If $x_{ij} = 1$ assign job j to m/c i

- $F := \{ (i,j) \mid 0 < x_{ij} < 1 \}$
- $G[F]$ has t conn. components.
- (m_t, n_t) be the # of m/c's & jobs in the t^{th} conn comp. F_t
- Just as in the matching case, each of these "grey" matrices must be full rank.



- Let's focus only when there is one comp. since we will repeat this for each.
- $\therefore \; |F_t| \le m_t + n_t$

$\Rightarrow$ $G[F_t]$ is a "pseudo-tree"



- No job has degree 1.
- There exists a "matching" assigning all jobs.
  - In the cycle go in any order
  - Shrink cycle to get a rooted tree
  - Assign job arb. to any child m/c
- $\boxed{\text{For all jobs } j, \text{ assign to machine matched.}}$

<u>Analysis</u> - Fix machine i
- It gets either j st $x_{ij} = 1$ or $\leq$ one job in step 2

$\therefore$ Total load on m/c $i \leq \sum_{j \in S_1} p_{ij} x_{ij} + p_{max}$

$\leq 2T$

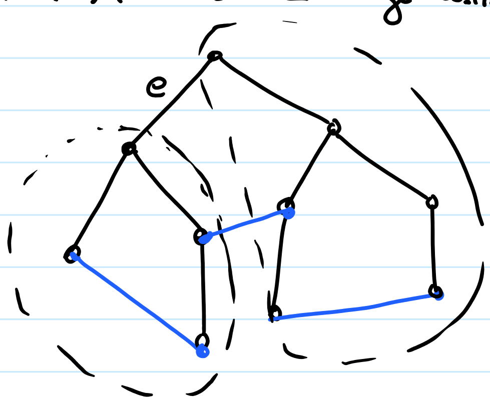
∎

Tree Augmentation Problem (TAP)

<u>Input</u> :-
- A rooted tree $T = (V, E)$
- Collection of $F \subseteq V \times V$ "links"
- Cost $c_f \;\; \forall f \in F$

<u>Output</u> : - $A \subseteq F$ s.t. $T + A$ is 2-edge-conn.

<u>Objective</u> :- Minimize $c(A)$



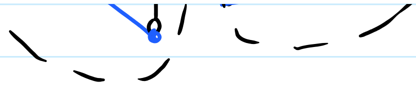
<u>LP-relaxation</u>

Min $\sum c_f x_f$

Min $\sum c_f x_f$

$\forall e \in T: \sum_{f \in F_e} x_f \geq 1$

$1 \geq x_f \geq 0$



$F_e := \{ f \in F : f$ has one ept in each tree $T_e \}$

<u>Thm</u>:- Let x be a bfs for the above LP. Then $\exists f : x_f \geq \frac{1}{2}$

Assuming the above theorem, TAP has a 2-appx
Pick f^* st $x_{f^*} \geq \frac{1}{2}$. This leads to a residual problem. (T', F') where T' is obtained by shrinking all edges e of T with $f \in F_e$ and deleting all f' which have epts among the edges deleted. Note, the old LP soln $x_f \ \forall f \in F'$ is a feas. soln.

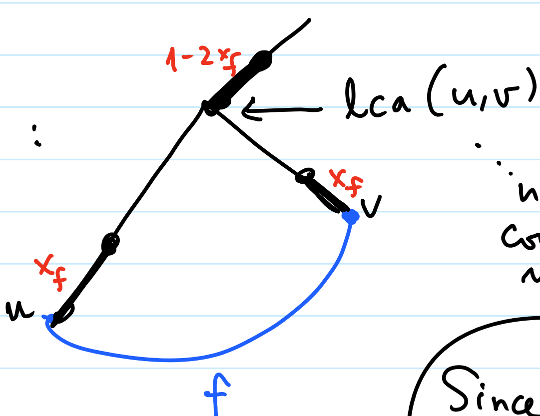
$$\therefore \quad LP(T,F) \geq LP(T',F') + \sum_{f \notin F'} c_f x_f$$
$$\geq LP(T',F') + \frac{1}{2} c_{f^*}$$

By "induction",
we can find $ALG(T',F') \leq 2 \cdot LP(T',F')$
$$ALG(T,F) = ALG(T',F') + c_{f^*}$$
$$\leq 2 LP(T,F)$$
∎

<u>Pf</u>:- Suppose not.

Clever Charging scheme:

For each $f = (u,v) \in F$, let $w \equiv lca(u,v)$



note this could be u or v

Put a charge on edge $(u, \text{parent}(u))$ of x_f
$(v, \text{parent}(v))$ " x_f

Since $x_f < \frac{1}{2}$ all this charge ...

edge $(u, \text{parent}(u))$ of x_f
$(v, \text{—"—}(v))$ " x_f
$(w, \text{—"—}(w))$ " $1-2x_f$

all this charge is >0

if $\text{parent}(w) = \emptyset$ $(w = r)$, then just lose the $1-2x_f$ charge ... and there must be some such.

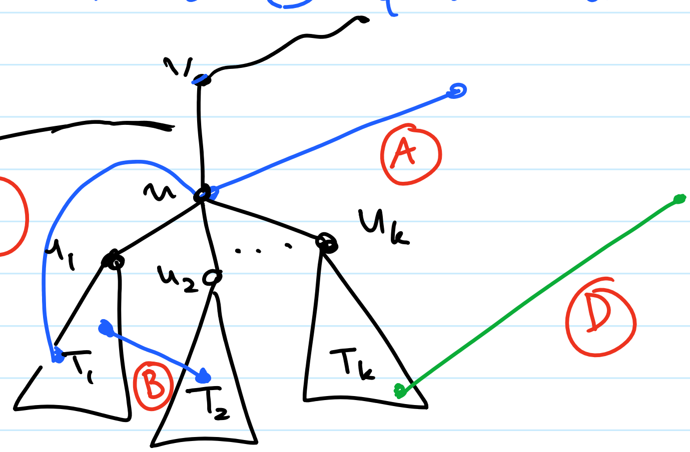
"Total charge distributed" $< |F| = \text{rank}(B)$ .... (*)

Let $B \subseteq E(T)$ be the set of edges achieving full rank

<u>Clm</u>: Each edge $e \in B$ get ≥ 1 unit of charge.

This would contradict (*) & we'll be done.

<u>Pf</u>:



where does it get charge from?

- Any link f from $T_i \to T_j$, $i \neq j$ $(1-2x_f)$
- Any link f from u "away" from the T_i's (x_f)
- Any link f from $T_i \to u$ $(1-x_f)$

① (u,v) gets > 0 charge.

If not, then $F_{uv} = F_{u,u_1} \cup F_{u,u_2} \cup \cdots \cup F_{u,u_k}$

$\Rightarrow$ The row-constraint corr. to (u,v) is .

$\equiv$ Sum of rows corr to (u, u_i)'s.

$\Rightarrow (u,v) \notin B.$ $\rightarrow\!\leftarrow$

② <u>(u,v) gets integer charge</u>.

$A : \{ f \mid f \in F_{u,v}$ & one ept is $u \}$

$B : \{ f \mid :$ one ept in T_i, other in T_j, $i \neq j \}$

$C : \{ f \mid f \in F_{u,u_i}$: with ept in $u \}$

$D : \{ f \mid f \in F_{u,v}$, orig from some $T_i \}$

$$x(F_{u,v}) = 1 \Rightarrow$$

$$x(A) + x(D) = 1$$

$$\sum_i x(F_{u,u_i}) = \sum_i 1 \Rightarrow$$

$$k = x(D) + 2x(B) + x(C)$$

Charge on (u,v)

$$\equiv \sum_{e \in B} (1 - 2x_e) + x(A) + \sum_{e \in C} (1 - x_e)$$

$$= |B| - 2x(B) + x(A) + |C| - x(C)$$

$$= |B| + |C| + 1 - (x(D) + 2x(B) + x(C))$$

$$= |B| + |C| + 1 - k \in \mathbb{Z}$$