Lecture 8

Wednesday, April 19, 2017 3:18 PM

Set Balancing

Input: Sets $S_1, S_2, \ldots, S_m \subseteq [n]$

Output: $\sigma : [n] \to \{-1, 1\}$ / {red, blue}

Objective:- → Find a signing which is as balanced as possible

Formally,

$$\boxed{\text{minimize} \quad \max_{i=1}^{m} \left| \sum_{j \in S_i} \sigma(j) \right|}$$

Quantity is called the Discrepancy of the Set-syst.

(*) This lecture we will work with the case when any element is in at most d - sets

• Casting as an Integer Program :

$$\text{Min} \quad T$$

$$\sum_{j \in S_i} \sigma_j \leq T \quad , \quad \forall i = 1 \ldots m$$

$$\sum_{j \in S_i} \sigma_j \geq -T \quad , \quad \forall i = 1, \ldots, m$$

$$\sigma_j \in \{-1, 1\}$$

$$\downarrow$$

$$-1 \leq \sigma_j \leq 1$$

→ What is the value of T? $T = 0$ when $\sigma = 0$

Algorithm

→ Maintain a coll^n of "safe" sets, $A \overset{\text{init}}{=} \emptyset$

→ Maintain a coll^n of "set" vars, $I \overset{\text{int}}{=} \emptyset$

// Ultimately $I = [n]$

→ Define a "surplus" vector $s \in \mathbb{R}^m$, a coor for every set, $s \overset{\text{init}}{=}$ all-zeros.

→ Consider $LP(A, I)$:

$$\left\{ x \in [-1,1]^{[n]\setminus I} : \quad \sum_{j \in S_i \setminus I} x_j = s_i, \quad \forall i \notin A \right\}$$

un-safe sets.

→ x be a bfs of $LP(A, I)$

→ If: $x_j = -1$ or $+1$, $I = I \cup j$ & $\sigma_j = x_j$

$\boxed{\text{Case A}}$ & $\forall i : j \in S_j$, $s_i = s_i - x_j$

if $x_j = 1$, then the remaining sum should try to be -1, and v.v.

// Inv: $\forall i \notin A$, $\sum_{j \in S_i \setminus I} x_j + \sum_{j \in S_i \cap I} \sigma_j = 0$

→ Else If: For some set S_i, $|S_i \setminus I| \leq d$

$\boxed{\text{Case B}}$ then $A = A \cup i$

We proclaim this set is safe. Why?

Note:- $\sum x_i + \sum \sigma_i = 0$

$$- \sum_{j \in S_i \setminus I} x_j + \sum_{j \in S_i \cap I} \sigma_j = 0$$

$$\Rightarrow \left| \sum_{j \in S_i \cap I} \sigma_j \right| \leq \left| \sum_{j \in S_i \setminus I} x_j \right| < d$$

and since LHS is integer, $\leq d-1$

Ultimately, $j \in S_i \setminus I$ will get some σ_j's; but

$$\left| \sum_{j \in S_i \setminus I} \sigma_j \right| \leq d \quad \because |S_i \setminus I| \leq d$$

$$\therefore \left| \sum_{j \in S_i} \sigma_j \right| \underset{\Delta\text{-ineq}}{\leq} \left| \sum_{j \in S_i \cap I} \sigma_j \right| + \left| \sum_{j \in S_i \setminus I} \sigma_j \right|$$

$$\leq 2d - 1$$

$\blacksquare$

<u>Claim</u>:- The algorithm terminates with $\sigma_j \in \{-1, 1\}$ for all elements.

<u>Pf</u>:- It suffices to show one of case A or case B always occurs.

Suppose case (A) doesn't occur.

$$\therefore \left| \{ j \mid 0 < x_j < 1 \} \right| = \# \text{ of cols}$$

$$\begin{array}{c} \| \\ \text{rank}(\text{"constr. matrix"}) \\ \wedge\!\backslash \\ \# \text{ of rows} \end{array}$$

But each col. has $\leq d$ ones (Problem ...)

But each col. has $\leq d$ ones (Problem assumption)

Since # rows $\geq$ # cols,

<u>some</u> row i has $\leq d$ ones

i.e. $|S_i \setminus I| \leq d \Rightarrow$ Case (B) occurs

$\blacksquare$

<u>Claim</u> :- At the end we have a $(2d-1)$-balanced coloring / signing.

<u>Pf</u> :- The blue stuff above proves this for safe sets

- For un-safe sets, in-fact, we have perfect balance

$\blacksquare$

<u>Note</u> :- This is not a standard Appx Algo. We didn't compare ourselves with the "best-possible" balance. Rather we proved that ANY set-system with each element in at most d-sets has a $(2d-1)$-balanced coloring.

It turns out that the q^n of whether there is a coloring with "perfect balance", i.e., whether $\exists\, \sigma : [n] \to \{-1, 1\}$ st.

$$\forall i, \quad \sigma(S_i) = 0$$

is in P, that is, it can be solved in poly-time.

$\therefore$ If $OPT = 0$, there can be an $ALG = 0$

o/w $OPT \geq 1$, and our thm. gives a $(2d-1)$-factor alg.