# On Competitiveness in Uniform Utility Allocation Markets *

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**Abstract**

We call a market *competitive* if increasing the endowment of one buyer does not increase the equilibrium utility of another. We show every competitive *uniform utility allocation* market is a *submodular utility allocation* market, answering a question of Jain and Vazirani, (2007). Our proof proceeds via characterizing non-submodular fractionally sub-additive functions.

**Keywords:** Market Equilibrium; Submodular functions; Fractionally Sub-Additive Functions

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# 1 Introduction

In the past few years, there has been a surge of activity to design efficient algorithms for computation of market equilibrium [7, 12, 8, 6] which in turn have given insights into various properties of the markets themselves. Recently, Jain and Vazirani [13] proposed a syntactic study of markets by introducing a new class of markets called *Eisenberg-Gale markets* or simply EG markets. In 1959, Eisenberg and Gale [9] gave a convex program for obtaining the equilibrium when agents have linear utilities and their endowments are proportional (the Fisher model). Jain and Vazirani define an EG market as any market whose equilibrium is captured by a similar convex program.

[13] showed that this class captured many interesting markets including several variants of resource allocation markets defined by Kelly [14] to model TCP congestion control.

Jain and Vazirani studied various properties of EG markets of which one was competitiveness (they used the term competition-monotone instead). A market is called competitive if increasing the money of one agent cannot lead to increase in the equilibrium utility of some other agent. The main result of this paper is to provide a characterization of competitive markets in a class of Eisenberg-Gale markets called uniform utility allocation markets which were also introduced by [13].

The convex program capturing equilibria of EG markets maximizes the money weighted geometric mean of the utilities of buyers over all *feasible* utilities, which form a convex set. If the constraints on feasible utilities are just those which limit the total utility obtainable by any set of agents, the EG market so obtained is called an *uniform utility allocation (UUA) market*. UUA markets can be represented via a set-function called the *valuation function*, where the value of any subset of agents denotes the maximum utility obtainable by that set. If the valuation function is submodular, the market is called a *submodular utility allocation (SUA) market*. In fact, the Fisher example above turns out to be a SUA market.

In their paper, [13] prove that every SUA market is competitive and ask if there all competitive UUA markets are SUA. We answer the question in the affirmative. We do so by obtaining non-trivial properties of a broad class of non-submodular functions which might be of independent interest.

# 2 Preliminaries

Traditionally, a market consists of a set of agents $A$ and a set of divisible items $J$, and each agent $i$ has an utility $u_i : \mathbf{R^j} \to \mathbf{R}_+$ and comes to the market with an initial endowment $e_i \in \mathbf{R^j}$ of goods. The market sets prices $p : J \to \mathbf{R}_+$ and trade occurs in the market. A market equilibrium is given by $(x_i, p^*)$ where $x_i$ denotes the final bundle of goods assigned to agent $i$ and the prices are market clearing prices having the following properties: for every agent the prices his final bundle equals that of his initial endowment, and at the given prices every agent gets his optimal bundle. A fundamental theorem of Arrow and Debreu [1] is that under certain very broad assumptions, market equilibrium always exists.

A significant sub-class of these general markets are when the set $A$ consists of one seller and all others are buyers, where sellers have all the items and buyers have money, and the seller has utility only for money and buyers have utilities only for items. This class of markets are called Fisher markets[11] (see also [4]) and predate general Arrow-Debreu markets. In 1959, Eisenberg and Gale[9] showed that for the Fisher markets when all buyers have linear utility functions, there exists a convex program whose optimum captures the equilibrium utilities of the agents and whose Lagrangean dual variables capture the equilibrium prices of the items. Motivated by this, Jain and Vazirani[13] defined the following syntactic class of markets.

**Definition 1** *An EG market $\mathcal{M}$ with agents $[n]$ is one where the feasible utilities $u \in \mathbf{R}^{\mathbf{n}}_+$ of the agents can be captured by a polytope*

$$\mathcal{P} = \{\forall j \in J : \quad \sum_{i\in[n]} a_{ij} u(i) \le b_j \quad u(i) \ge 0\}$$

*with the following* free disposal *property: If $u$ is a feasible utility allocation, then so is any $u'$ dominated by $u$.*

**Examples**: The convex program of Eisenberg and Gale[9] which captures the equilibrium utilities of linear utility Fisher markets is precisely an EG market with

$$\mathcal{P} = \{\forall i,\ u_i = \sum_{j \in J} u_{ij} x_{ij}; \quad \forall j,\ \sum_{i \in A} x_{ij} \le 1\}$$

where $J$ is the set of items and $A$ is the set of agents. Another example is the following resource allocation market defined by Kelly. Given a network, agents own source-sink pairs and wish to buy capacities on edges so as to send flows from source to sink. The utility $u(i)$ of each agent is the amount of flow it sends. The various flow vectors are constrained via capacity constraints on each edge, which form the convex flow polytope $\mathcal{P}$ above.

**Remark**: The definitions in [13, 5] also include auxiliary variables (as in the example of linear utility Fisher above) in the definition, but the above is an equivalent definition which will be sufficient for the purposes of this paper.

An instance of an EG market $\mathcal{M}$ is given by the money of the agents $m \in \mathbf{R}^{\mathbf{n}}_{+}$. The equilibrium utility allocation of an EG market is captured by the following convex program similar to the one considered by Eisenberg and Gale [9] for the Fisher market with linear utilities.

$$\max \sum_{i=1}^{n} m_i \log u(i) \quad \text{s.t.} \quad u \in \mathcal{P}$$

Since the objective function is strictly concave and $\mathcal{P}$ is non-empty, the equilibrium always exists and is unique. Applying the Karash-Kuhn-Tucker (KKT) conditions (see e.g. [3]) characterizing optima of convex programs, for each constraint we have a Lagrangean variable $\mathbf{p_j}$ which we think of as *price* of the constraint, and we have the following equivalent definition of equilibrium allocations in EG markets.

**Definition 2** *Given a market instance $m \in \mathbf{R}^{\mathbf{n}}_{+}$ of an EG market $\mathcal{M}$, a feasible utility allocation $u \in \mathbf{R}^{\mathbf{n}}_{+}$ is an* equilibrium allocation *if there exists* prices $p \in \mathbf{R}^{|\mathbf{J}|}_{+}$ *satisfying*

- *For all agents $i \in [n]$, $m_i = u(i) \cdot rate(i)$ where $rate(i) = (\sum_{j \in J} a_{ij} p(j))$, the money spent by agent $i$ to get unit utility.*
- $\forall j \in J : p(j) > 0, \qquad \sum_{i \in [n]} a_{ij} u(i) = b_j$

Thus, in the equilibrium allocation, only those constraints are priced which are satisfied with equality (these constraints are called tight constraints), and each agent exhausts his or her money paying for the utility he obtains.

**Example** In the case of the Fisher setting with linear utilities, there is a constraint for each good and the prices exactly correspond to the unit price of the good. In the resource allocation market described above, there is a price for each edge. Price of an edge is non-zero only if it is saturated by the various flows and each agent exhausts his or her money buying the capacities on edges.

We now consider the case when each $a_{ij}$ above is either 0 or 1.

**Definition 3** *An EG market $\mathcal{M}$ is a UUA market if the feasible region $\mathcal{P}$ of utilities can be encoded via a* valuation function $v : 2^{[n]} \to \mathbf{R}$ *as follows*

$$\mathcal{P} = \{\forall S \subseteq [n] \qquad \sum_{i \in S} u(i) \leq v(S)\}$$

*Such an EG market will be denoted as $\mathcal{M}(v)$, as the market constraints is completely described by $v$.*

**Definition 4** *If the valuation function $v$ in Definition 3 is a submodular function, then the market is called a* Submodular Utility Allocation *(SUA) market. To remind, a function $v : 2^{[n]} \to \mathbf{R}$ is submodular if for all sets $S, T \subseteq [n]$, $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$.*

**Example** It is not hard to see if in the resource allocation market above, all the agents have the same source, then the coefficients in the flow polytope description are all 0 or 1, implying the market is UUA. In fact, [13] show that this market is SUA as well.

For UUA (and SUA) markets, as in Definition 2 the following gives a characterization of the equilibrium allocation. Given a feasible utility allocation $u$, a set $S$ is called *tight* if $u(S) \equiv \sum_{i \in S} u(i) = v(S)$.

**Definition 5** *For a UUA market, an utility allocation $u$ is the equilibrium allocation iff there exists prices for each subset $S \subseteq [n]$ such that*

- $\forall S \subseteq [n], p(S) > 0 \Rightarrow S$ *is tight.*
- *For all $i \in [n]$, $m_i = u(i) \cdot rate(i)$ where $rate(i) = \sum_{i \in S} p(S)$.*

Given a UUA market, the following observation of [13] shows assumptions we can make on the valuation function. For completeness, we give a proof as well.

**Lemma 2.1** *([13]) The valuation function $v$ of UUA markets can be assumed to have the following properties*

- Non degeneracy*: $v(\varnothing) = 0$*
- Monotonicity*: $S \subseteq T \Rightarrow v(S) \leq v(T)$*
- Non redundancy of sets*: For any subset of agents $T \subseteq [n]$, there exists a feasible utility allocation $u$ such that $\sum_{i \in T} u(i) = v(T)$.*
- Complement free*: $v(S \cup T) \leq v(S) + v(T)$.*

**Proof:** Given the valuation function $v$ define the *covering closure* $v^*$ of $v$ as the follows. For a set $S \subseteq [n]$,

$$v^*(S) = \{\min \sum_{T \subseteq [n]} v(T) x_T \ \text{ s.t } \ \forall i \in S \sum_{T : i \in T} x_T \geq 1 \quad x_T \geq 0\} \tag{1}$$

We prove $v^*$ satisfies all the properties above and $v^*$ is feasible iff $v$ is feasible.

It is easy to see from definition that $v^*$ is non degenerate and monotone. By LP duality we have the following characterization of $v^*$. For any set $S \subseteq [n]$,

$$v^*(S) = \{\max \sum_{i \in S} u(i) \ \text{ s.t } \ \forall T \subseteq [n] \sum_{i \in T} u(i) \leq v(T) \quad \forall i, \ u(i) \geq 0\}$$

We deliberately use the variable $u(i)$ in the dual. Note that any feasible allocation for $v$ is a feasible solution to the above. This shows that if $u$ is feasible for $v$, then it is also feasible for $v^*$.

Non redundancy follows from the fact that for each set $S$, the dual variables corresponding to the above program gives a feasible allocation with the property $v^*(S) = \sum_{i \in S} u(i)$. Complement free follows from non redundancy as follows: there exists feasible allocation $u$ such that $v^*(S \cup T) = \sum_{i \in S \cup T} u(i) \leq \sum_{i \in S} u(i) + \sum_{i \in T} u(i) \leq v^*(S) + v^*(T)$, where the last inequality follows via feasibility. $\Box$

Set functions which have the form as in Equation(1) are called *fractionally sub-additive*. This forms a rich class of set functions and arise various forms in economics and game theory[2, 15, 10]. Lemma 2.1 says that the valuation function of EG markets can be assumed to fractionally sub-additive.

We now define competitiveness.

**Definition 6 ([13])**
*An EG market $\mathcal{M}$ is* competitive *(competition monotone in [13]) if for any money vector $m$, any agent $i \in [n]$ and all $\epsilon > 0$, let $u, u'$ be the equilibrium allocations with money $m$ and $m'$, where $m'(j) = m(j)$ for all $j \neq i$ and $m'(i) = m(i) + \epsilon$, we have $u'(j) \leq u(j)$ for all $j \neq i$.*

In Section 3, we prove the main result of this paper.

**Theorem 2.2** *If a UUA market is competitive, then it is an SUA market.*

# 3 Competitive $UUA$ markets are $SUA$ markets

In this section we prove Theorem 2.2. First we state a property we need for non-submodular functions, which we prove in Section 4.

**Theorem 3.1** *Given any valuation function $v$ which is fractionally sub-additive (satisfying the conditions of Lemma 2.1) which is not submodular, there exists set $T, i, j$ and a feasible utility allocation $u$ such that*

1. *$T, T \cup i, T \cup j$ are tight.*
2. No set *containing both $i$ and $j$ is tight*
3. *All tight sets containing either $i$ or $j$ contain a common element $l$ with $u(l) > 0$.*

**Proof of Theorem 2.2 :** Let $\mathcal{M}$ be any UUA market which is not an SUA market. We construct money vectors $m_1$ and $m_2$ along with the respective equilibrium utility allocations $u_1$ and $u_2$, with the following properties:

- $m_2(i) \geq m_1(i)$ for all $i \in [n]$
- There exists $j$ with $m_2(j) = m_1(j)$ and $u_2(j) > u_1(j)$

We first show the above contradicts competitiveness. Since $m_2$ is greater than $m_1$ in each coordinate, we can construct vectors $m'_1, m'_2, \cdots, m'_k$ for some $k$, such that $m'_1 = m_1$, $m'_k = m_2$ and each consecutive $m'_i, m'_{i+1}$ differ in exactly one coordinate $j'$ with $m'_{i+1}(j') > m'_i(j')$. Note that $m'_i(j) = m_1(j) = m_2(j)$.

Let $u'_1, u'_2, \cdots, u'_k$ be the equilibrium allocations corresponding to the money vectors. We have $u_1 = u'_1$ and $u_2 = u'_k$. $u_2(j) > u_1(j)$ implies for some consecutive $i, i+1$ also $u'_{i+1}(j) > u'_i(j)$. Since $m'_{i+1}(j) = m'_i(j)$, we get the contradiction.

To construct the vectors $m_1, m_2$, we need the structural theorem 3.1. Let $T, i, j, l, u$ be as in the theorem. To construct both the instances, we first construct feasible utilities and then derive the money vectors such that the allocation are indeed equilibrium utility allocations.

Let $u_1 := u$ except $u_1(i) = 0$. Define $m_1(k) = u_1(k)$ for all $k$. By condition 1 in Theorem 3.1, we get $T \cup j$ is tight. Pricing $p(T \cup j) = 1$ shows $u_1$ is the equilibrium allocation with respect to $m_1$.

Let $u_2 := u$ except $u_2(i) = u(i) + \epsilon$, $u_2(j) = u(j) + \epsilon$ and $u_2(l) = u(l) - \epsilon$ for some $\epsilon > 0$. $\epsilon$ is picked to satisfy two properties: $(a)\epsilon \le u(l)/2$ and $(b)u_2$ is feasible. We show later how to pick $\epsilon$. Construct $m_2$ as follows. Define $p' := u_1(j)/u_2(j)$. $m_2(j) = m_1(j)$, $m_2(k) = (2 + p')u_2(k)$ for all $k \in T$, and $m_2(i) = u_2(i)$. Check that $m_2$ dominates $m_1$ in each coordinate and $m_2(j) = m_1(j)$.

To see $u_2$ is an equilibrium allocation w.r.t $m_2$, note that $T \cup i$, $T \cup j$ remain tight. Let $p(T \cup i) = 2$, $p(T \cup j) = p'$. Check all the conditions of Definition 5 are satisfied.

The proof is complete via the definition of $\epsilon$. Note that in the allocation $u_2$, the sets which have more utility than in $u$ are ones which contain $i$ or $j$. By conditions of Theorem 3.1, one can choose $\epsilon$ small enough so that $u_2$ doesn't make any new set tight and is smaller than $u(l)/2$. To be precise, let

$$\begin{aligned}
\epsilon_i &:= \min_{Z \subseteq T : Z \cup i \text{ not tight}} (v(Z \cup i) - u(Z \cup i)) \\
\epsilon_j &:= \min_{Z \subseteq T : Z \cup j \text{ not tight}} (v(Z \cup j) - u(Z \cup j)) \\
\epsilon_{ij} &:= \min_{Z \subseteq T} \frac{v(Z \cup i \cup j) - u(Z \cup i \cup j)}{2}
\end{aligned}$$

Note by definition $\epsilon_i, \epsilon_j > 0$, and by condition 3 above, $\epsilon_{ij} > 0$. Choose $\epsilon := \min(\epsilon_i, \epsilon_j, \epsilon_{ij}, u(l)/2)$. Again $\epsilon > 0$ which completes the proof of Theorem 2.2. □

# 4 A property of non-submodular fractionally sub-additive set functions

In this section we prove the technical theorem 3.1.

**Proof of Theorem 3.1 :** Since $v$ is not submodular, there exists a set $S \cup j$ contradicting submodularity. That is, there is a set $S$, an agent $j \notin S$ and a strict subset $T \subsetneq S$, such that the marginal value of $j$ for $S$ is greater than that for $T$. That is, $v(S \cup j) - v(S) > v(T \cup j) - v(T)$

Choose $S \cup j$ to be the smallest set contradicting submodularity. Choose $T$ to be the subset of $S$ for which $v(T \cup j) - v(T)$ is minimum. $T, j$ are that of the theorem.

Since $S$ is the smallest, the restriction of $v$ to every subset of $S \cup j$, in particular $T$ is submodular. Since $T$ is chosen to make marginal of $j$ the minimum, $T$ would have cardinality exactly 1 less than that of $S$. That is $S = T \cup i$. This is the $i$ in the theorem. Note, we have

$$v(T \cup i \cup j) - v(T \cup i) > v(T \cup j) - v(T) \tag{2}$$

Since $v$ satisfies the non-redundancy condition, there exists a feasible allocation $u$ which tightens $T$. For any $u$, define the family of tight subsets of $T$ as $\mathcal{F} = \{Z \subseteq T : u(Z) = v(Z)\}$. Note that $\mathcal{F}$ is nonempty since $T \in \mathcal{F}$. Choose $u$ so that $|\mathcal{F}|$ is minimum. Define $u(j) := v(T \cup j) - v(T)$ and $u(i) := v(T \cup i) - v(T)$. This completes the definition of $u$ of the theorem.

We now prove feasibility of $u$ and the properties 1,2,3 in the statement of the theorem. We need the following structural facts which we prove later.

**Lemma 4.1** *$\mathcal{F}$ is closed under taking complements, that is, if $Z \in \mathcal{F}$, so is $T \setminus Z$*

**Lemma 4.2** *Let $v$ restricted to a set $X$ be submodular. The set of tight subsets of $X$ are closed under taking unions and intersections.*

**Lemma 4.3** *Union of disjoint tight sets is tight.*

We first show that $u$ is feasible. Lemma 4.4 shows the feasibility for sets containing either $i$ or $j$. Lemma 4.5 shows the feasibility for sets containing both, and in fact proves Property 2. This is sufficient byb definition. Property 1 of the theorem follows directly from definition of $u$. We prove property 3 after these two lemmas.

**Lemma 4.4** *$u$ is feasible over $T \cup i$ and $T \cup j$. In fact, if $Z \cup i$ or $Z \cup j$ is tight, then so is $Z$.*

**Proof:** Pick any subset $Z \subseteq T$. We get $u(Z \cup i) = u(Z) + v(T \cup i) - v(T)$. Since $v$ restricted to $T \cup i$ is submodular, we get $u(Z \cup i) \leq u(Z) + v(Z \cup i) - v(Z) \leq v(Z \cup i)$. Thus $u$ is feasible over $T \cup i$. Also, if $Z \cup i$ were tight, we would have $u(Z) = v(Z)$ implying $Z$ were tight. $\Box$

**Lemma 4.5** *For all subsets $Z \subseteq T$, $u(Z \cup i \cup j) < v(Z \cup i \cup j)$.*

**Proof:** Pick any set $Z$. Note that

$$u(Z \cup i \cup j) = u(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T) \tag{3}$$

Note that the union of sets $(Z \cup i \cup j) \cup (T \setminus Z) = T \cup i \cup j$. Thus by the complement free condition of $v$, we get

$$v(Z \cup i \cup j) \geq v(T \cup i \cup j) - v(T \setminus Z)$$

Two cases arise. Suppose $Z \in \mathcal{F}$, that is, $Z$ is tight. We take care of the other case later. Then, by Lemma 4.1, $T \setminus Z$ is also tight. Thus, $v(Z) + v(T \setminus Z) = v(T)$ since all the sets are tight. Putting this in above equation and applying Equation 2 we get $v(Z \cup i \cup j) \geq v(T \cup i \cup j) - v(T) + v(Z) > v(T \cup i) + v(T \cup j) - v(T) - v(T) + v(Z) \geq u(Z \cup i \cup j)$ from equation 3.

Now suppose $Z \notin F$, that is, $u(Z) < v(Z)$. Thus Equation 3 implies

$$u(Z \cup i \cup j) < v(Z) + v(T \cup i) - v(T) + v(T \cup j) - v(T)$$

By submodularity of $T \cup i$, we get $v(T \cup i) - v(T) \leq v(Z \cup i) - v(Z)$. Also, by choice of $T$ to be the subset of $T \cup i$ minimizing $v(T \cup j) - v(T)$, we get $v(T \cup j) - v(T) \leq v(Z \cup i \cup j) - v(Z \cup i)$. Plugging this in the equation above proves the lemma. $\Box$

To prove property 3, we make a few definitions. Analogous to $\mathcal{F}$, define $\mathcal{F}_i := \{Z \subseteq T : u(Z \cup i) = v(Z \cup i)\}$. Similarly define $\mathcal{F}_j$. By Lemma 4.4, all sets in $\mathcal{F}_i$ and $\mathcal{F}_j$ are tight. Property 3 is implied by $\bigcap_{Z \in \mathcal{F}_i, \mathcal{F}_j} Z$ contains an element $l$ with $u(l) > 0$.

Lemma 4.6 shows no two sets in $\mathcal{F}_i$ or $\mathcal{F}_j$ are disjoint. By Lemma 4.2, this implies the tight sets $T_i := \bigcap_{Z \in \mathcal{F}_i} Z$ (similarly $T_j$) are non-empty. Lemma 4.7 shows that $T_i$ and $T_j$ are not disjoint and in fact $v(T_i \cap T_j) > 0$ which by tightness of $T_i \cap T_j$ proves implies existence of $l$.

**Lemma 4.6** *No two sets in $\mathcal{F}_i$ or $\mathcal{F}_j$ are disjoint.*

**Proof:** We prove for $\mathcal{F}_j$, that for $\mathcal{F}_i$ is similar. Suppose there existed $A, B \in \mathcal{F}_j$ disjoint. Since $A \cup j$ and $B \cup j$ are tight, and $v$ is submodular when restricted to $T \cup j$, we get the intersection of $A \cup j$ and $B \cup j$, the set $j$, is tight. Since $T \cup i$ is tight by Condition 1, we get $T \cup i \cup j$ is tight, contradicting Condition 2, which is already proven. $\Box$

**Lemma 4.7** $v(T_i \cap T_j) > 0$

**Proof:** Since $T_i$ and $T_j$ are tight, so is $T_i \cup T_j$ and by Lemma 4.1, so is $T \setminus (T_i \cup T_j)$. If $T_i$ and $T_j$ were disjoint, then $T \cup i \cup j$ is the disjoint union of the tight sets $(T_i \cup i), (T_j \cup j)$ and $T \setminus (T_i \cup T_j)$. By lemma 4.3, $T \cup i \cup j$ is tight contradicting Condition 2. Note that we cannot use Lemma 4.2 as $v$ is *not* submodular when restricted to $T \cup i \cup j$, and need disjoint condition. $\Box$ $\Box$

## 4.1 Proofs of facts used in Theorem 3.1

**Proof of Lemma 4.2 :** Let $A, B \subset X$ be two tight subsets of $X$. Since $u$ is a feasible allocation, Thus we get

$$u(A) + u(B) = u(A \cup B) + u(A \cap B)$$
$$\leq v(A \cup B) + v(A \cap B) \leq v(A) + v(B) = u(A) + u(B)$$

implying equality throughout. In particular, $u(A \cap B) = v(A \cap B)$ and $u(A \cup B) = v(A \cup B)$. □

**Proof of Lemma 4.3 :** For any disjoint sets $A, B$, $u(A \cup B) = u(A) + u(B)$ and thus $v(A \cup B) \leq v(A) + v(B) = u(A) + u(B) = u(A \cup B) \leq v(A \cup B)$ where the first inequality follows from complement-free condition on $v$. □

**Proof of Lemma 4.1 :** Suppose $T \setminus Z$ is not tight. We modify $u$ so that no new set becomes tight and $Z$ also becomes untight. This contradicts the minimality of $\mathcal{F}$. Call $T \setminus z$ as $X$.

Pick an element $x \in X$. Let $A$ be the *smallest* tight set containing $x$. This is defined since $T$ contains $x$. We might assume $x$ is picked so that $A$ intersects $Z$. If no such $x$ existed, then $X$ is a union of tight sets and we are done by Lemma 4.2.

Denote $A \cap Z$ by $Y$. Let $y \in Y$ be with $u(z) > 0$. We prove such a $y$ exists shortly. Since $A$ was the smallest tight set containing $x$, all tight sets containing $x$ also contains $A$. Therefore, modifying $u$ to give suitable small more utility to $x$ and exactly that less to $y$ renders it feasible and leaves both $Z$ and $T \setminus Z$ untight.

To see the existence of $y \in Y$ with $u(y) > 0$, note that if not then we get $u(Y) = 0$. Thus $v(A) = u(A) = u(A \cap X) \leq v(A \cap X) \leq v(A)$, implying $A \cap X \subsetneq A$ is also tight. Thus contradicts the minimality of $A$. □

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