Submodularity Helps in Nash and Nonsymmetric Bargaining Games[*]

Deeparnab Chakrabarty[†]   Gagan Goel[‡]   Vijay V. Vazirani [§]   Lei Wang[¶]

Changyuan Yu[‖]

Abstract

Motivated by the recent work of [Vaz12], we take a fresh look at understanding the quality and robustness of solutions to Nash and nonsymmetric bargaining games by subjecting them to several stress tests. Our tests are quite basic, e.g., we ask whether the solutions are computable in polynomial time, and whether they have certain properties such as efficiency, fairness, desirable response when agents change their disagreement points or play with a subset of the agents.

Our main conclusion is that imposing *submodularity*, a natural economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties.

[†]Current address: Microsoft Research, Bangalore, India 560001. Email: dechakr@microsoft.com. Work done as a graduate student at Georgia Tech.

[‡]Current address: Google Inc., NYC, New York. Email: gagan@google.com. Work done as a graduate student at Georgia Tech.

[§]College of Computing, Georgia Tech, Atlanta, GA 30332–0280. Email: vazirani@cc.gatech.edu

[¶]Current Address: Microsoft Corporation, Bellevue, WA 98004 Email: leiwa@microsoft.com. Work done as a graduate student at Georgia Tech.

[‖]Institute for Theoretical Computer Science, Tsinghua University

1 Introduction

Bargaining is perhaps the oldest situations of conflict of interest, and since game theory develops solution concepts for negotiating in such situations, it is perhaps not surprising that bargaining was first modeled as a game in John Nash's seminal 1950 paper [Nas50], using the framework of game theory given a few years earlier by von Neumann and Morgenstern [vNM44]. Since then, this has led to a theory (of bargaining) that lies today at the heart of game theory (e.g., see [Kal85, TL89, OR90, Aum94]).

In Nash's model, n agents participating in the bargaining game define a feasible set of utilities depending on the constraints of the game. Furthermore, each agent also possesses a *disagreement utility* below which she pulls out of the game. The feasible set is assumed to downward closed, and the n-tuple of disagreement utilities are assumed to lie in the set. Nash proposed that a solution to the bargaining game must satisfy four reasonable axioms, and proved that there is a unique point in the feasible set of utilities of the agents which satisfies all these.

Over the years, the theory of bargaining developed along two major aspects. The more prominent one was the axiomatic approach whose goal was to characterize the solution concept that results from assuming a given set of axioms. Notable is the work of Kalai [Kal77] which removes one of the axioms of Nash (that of symmetry). Unlike the Nash bargaining solution, which is unique, a game can have infinitely many nonsymmetric bargaining solutions. One can single out one of these by specifying the *clout* of each player. Kalai's main theorem implies that the nonsymmetric solution to an n-person bargaining game, with integral clouts, corresponds to a Nash bargaining solution to a game with a larger number of players which is obtained by replicating each of the n players as many times as her clout.

The second direction was to determine the quality of the basic solution concepts of this theory by subjecting them to what may be viewed as 'stress tests'. These determine whether or not a particular solution concept satisfies a particular property on *all instances* of the bargaining game. We list the properties of Nash bargaining games that have been studied so far (e.g., see [Tho08]). Thomson [Tho87] shows that all such games satisfy *monotonicity*, i.e., if the disagreement utility of a player is increased, then her utility cannot decrease. He also studied *strong monotonicity*, i.e., if the disagreement utility of one player is increased, then any other player's utility should not increase. He showed that Nash bargaining games do not always satisfy strong monotonicity. Thomson [Tho83] also defines the notion of *population monotonicity*, i.e., when bargaining with a proper subset of the players, an agent's utility should be non-decreasing, and shows that the Nash Bargaining solution doesn't satisfy this property.

Although useful, such 'negative' results can give only limited information on the quality of a solution concept. In particular, the instances of interest may be highly structured and therefore may pass the test even though the entire solution concept does not. Motivated by recent insights on the two solution concepts of Nash and nonsymmetric bargaining games, obtained from an algorithmic study of these notions [Vaz12], we take a fresh look at understanding their quality and robustness. We deviate from the earlier line of work in two respects. First, we use the classification of these games given in [Vaz12], to determine if specific classes of these games satisfy a given property. Second, we define and study new tests that follow naturally from an algorithmic game theory perspective.

Our main conclusion is that imposing *submodularity*, a natural and well-studied economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties. More precisely, if the maximum possible feasible utility obtained by a subset of agents is

bounded by a submodular function on the subset, then the corresponding Nash and nonsymmetric bargaining solutions satisfy all the properties we study. Our new tests are of a basic nature. The first test is computational: whether the Nash and nonsymmetric bargaining solutions can be efficiently computed in polynomial time. We show that submodularity leads to combinatorial, strongly polynomial time algorithms. Then we ask whether the solutions yielded by these two concepts are "efficient" and "fair", after giving reasonable, quantitative definitions of efficiency and fairness. We also define and study the notion of *surplus monotonicity*, i.e., if the disagreement utility of a player is increased, then her surplus utility, over and above the disagreement utility, increases. As stated above, an important catalyst for our results is the insights gained from the *combinatorial* algorithmic approach to the Nash and nonsymmetric bargaining games.

1.1   Detailed description of our results

In order to describe our results more formally, we need to recall some classes of Nash and nonsymmetric bargaining games defined recently in [Vaz12]. LNB (Linear Nash and Nonsymmetric Bargaining Games) is the class of games whose feasible set of utilities is defined by finitely many linear constraints. UNB (Uniform Utility Nash and Nonsymmetric Bargaining Games) is the subclass of LNB in which for each available resource, each agent who uses this resource uses it in the same way, i.e., the linear constraints are all packing constraints and the coefficients in each constraint are $0/1$. Clearly, only $2^{|A|}$ such constraints are needed, where A is the set of agents – one for each subset of A. We can now view the right hand sides of these constraints as being given by a set valued function over the power set of A. If this function is submodular, the game is said to be in the subclass SNB (Submodular Utility Nash and Nonsymmetric Bargaining Games) of UNB.

UNB and SNB capture a large class of bargaining scenarios. For the interested reader, at the risk of violating linearity of exposition, we refer to Section 3 for several illustrative examples of UNB and SNB games. We give one simple example here as it might be useful to keep it in mind for the forthcoming properties. Consider n agents bargaining for k divisible items. The utility of an item j to any agent is either 0 or a fixed quantity p_j (say the *value* of item j) that depends only on item j. Furthermore the utilities of all agents are linear. It is not too hard to see that the feasible set of utilities form a downward closed polytope. In fact, given a subset S of agents, the total utility obtainable by this subset is precisely the total value of items which give non-zero utility to at least one agent in the set; and these inequalities capture the polytope. Thus, this bargaining game lies in UNB, and in fact it lies in SNB (since neighborhood functions are submodular).

The first result of our paper is the computational complexity of the solutions in SNB. We give a completely combinatorial, strongly polynomial time algorithm to compute solutions for all games in SNB; as a corollary we show that equilibrium allocations are always rational numbers. Furthermore, the insights obtained in developing this algorithm is useful in our subsequent results regarding the various properties of bargaining games.

Several of our results show that a game in UNB has a certain property iff it is in SNB. Equivalently, these results characterize SNB within UNB. These results should not come as a surprise; submodularity has been exploited in similar ways in the past in game theory, e.g., see Moulin and Shenker [MS01], showing that a cross-monotonic cost sharing method gives rise to a budget balanced and group strategyproof cost sharing mechanism if the cost function is submodular, and the results of [JV10],[CD09], giving analogous characterizations of Eisenberg-Gale markets defined via submodular functions. We now describe the properties we study; the reader may want to keep the example described above in mind for an illustration.

We say that a Nash bargaining game is *utilitarian efficient* if and only if the total value of the Nash bargaining solution is the same as that of the most efficient solution that can be obtained in a centralized manner. We will say that a Nash bargaining game is *strongly monotone* if whenever one player increases his disagreement utility, no other player's utility can increase in the resulting Nash or nonsymmetric bargaining solution. We define the *surplus utility* of an agent to be the utility she gets in a Nash or nonsymmetric bargaining solution, over and above her disagreement utility. We say that a Nash or nonsymmetric bargaining game is *surplus monotone* if increasing the disagreement utility of any agent does not lead to an increase in her surplus utility. We say that a bargaining solution is *min-max fair*, if the surplus utility vector of the bargaining solution sorted in a decreasing order is lexicographically dominated by the surplus utility obtained from any other Pareto optimal solution; a solution is Pareto optimal if there is no other feasible utility coordinate-wise dominating the same. Similarly, *max-min* fairness is defined.

The above properties fixed the number of players involved in the game. The following notion, defined by Thomson [Tho94], studies the quality of the solution as the set of agents change. A Nash bargaining game is defined to be *population monotone* if the utility of any agent in the bargaining solution can only increase when playing with a subset of the agents. Similarly, we say that a nonsymmetric bargaining game is *population monotone* if the utility of any agent in the bargaining solution can only increase if clouts of other agents are decreased.

For each of the properties stated above, except surplus monotonicity, we show that a game in UNB has this property iff it is also in SNB, i.e., *these properties characterize SNB within UNB.*

Organisation. We start with formal definition of Nash and nonsymmetric bargaining games, UNB, SNB, and the properties described above, in Section 2. We then give illustrative examples in Section 3. After giving some mathematical preliminaries in Section 4, we spend the subsequent sections describing each property we study.

2   Nash and Nonsymmetric Bargaining Games

For a set of agents A, a Nash bargaining game is defined by a pair[1] $(\mathbf{c}, \mathcal{P})$, where $\mathcal{P} \subseteq R_+^{|A|}$ is a compact and convex set which defines the feasible set of utilities of all the agents, and $\mathbf{c} \in \mathcal{P}$ is known as the disagreement point which defines the amount of utility each agent will get if the bargaining process fails.

Nash [Nas50] defined the bargaining solution $\mathbf{u}^* \in \mathcal{P}$ of this game to be the one which satisfies four axioms:

1. **Pareto optimality**: No point x in $\mathcal{P}$ can weakly dominate $\mathbf{u}^*$, that is, $x_i \geq u_i^*$ for all $i \in A$.

2. **Invariance under transformation of utilities**: The affine transformation of the utilities leads to the same affine transformation on $\mathbf{u}^*$.

3. **Symmetry**: If the agents are renumbered, then it suffices to renumber the coordinates of $\mathbf{u}^*$.

4. **Independence of irrelevant alternatives**: If $\mathbf{u}^*$ is the solution for $(\mathbf{c}, \mathcal{P})$ where $\mathbf{c}, \mathbf{u}^* \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{P}$, then $\mathbf{u}^*$ is also the solution for $(\mathbf{c}, \mathcal{S})$.

---

[1]Throughout the paper, we will use boldface to denote vectors, however, when we wish to denote the coordinates we will revert to normal font. For instance, $\mathbf{c}$ is a vector, c_i is the ith coordinate, which is a scalar

Nash proved that there is a unique point in $\mathcal{P}$ which satisfies these axioms, and moreover this point ($\mathbf{u} \in \mathcal{P}$) is the one that maximizes $\prod_{i \in A}(u_i - c_i)$ or equivalently $\sum_{i \in A} \log(u_i - c_i)$.

In the nonsymmetric bargaining game, each agent i has a positive bargaining weight or clout α_i which represents his relative bargaining power. The bargaining solution in this case is the point in the feasible region $\mathcal{P}$ which maximizes the objective function $\sum_{i \in A} \alpha_i \log(u_i - c_i)$.

2.1 Uniform Utility Nash and Nonsymmetric Bargaining Games

The class Linear Nash and Nonsymmetric Bargaining Games (LNB), defined in [Vaz12], consists of games whose feasible set $\mathcal{P}$ is defined by a finite number of linear constraints. The main focus of our paper will be on the subclass of LNB called Uniform Utility Nash and Nonsymmetric bargaining games (or UNB) [Vaz12]. In these games, the linear constraints are all packing constraints and in each of them, the coefficients of the variables are either 0 or 1. Clearly there can be at most $2^{|A|}$ such constraints, thus a function of the form $v : 2^A \to R^+$ uniquely encodes a feasible set in UNB games. A UNB game is called an SNB if the function v is a submodular function. As stated in the Introduction, we give examples of UNB and SNB games in Section 3.

Now given a disagreement point $\mathbf{c}$, and a fixed set of agents $T \subseteq A$, the solution to a uniform utility Nash bargaining game among the agents in T is captured by the following convex program:

$$
\begin{aligned}
\max \quad & \sum_{i \in T} \log(u_i - c_i) && (1) \\
s.t. \quad & \forall S \subset T : \sum_{i \in S} u_i \leq v(S) \\
& \forall i \in T : u_i \geq 0
\end{aligned}
$$

For uniform utility nonsymmetric bargaining games, the objective functions that needs to be maximized is $\sum_{i \in T} \alpha_i \log(u_i - c_i)$.

For a fixed function $v : 2^A \to R^+$, we will define a family of games $F(v)$ to be the set of all Nash bargaining games for various choices of disagreement points $\mathbf{c}$ and sets $T \subseteq A$. An instance $(\mathbf{c}, T) \in F(v)$ will refer to a particular Nash bargaining game in $F(v)$ with a fixed set T and disagreement point $\mathbf{c}$. We call a game $(\mathbf{c}, T)$ *feasible* if there exists at least one feasible $\mathbf{u}$ with $u_i > c_i$ for all $i \in T$. Otherwise, we call it *infeasible*. We will use $u_i^*(\mathbf{c}, T)$ to denote the utility of the player i in the Nash's solution for a feasible bargaining game $(\mathbf{c}, T) \in F(v)$.

Similarly, for a fixed function $v : 2^A \to R^+$ and a positive bargaining weight vector $\alpha = (\alpha_i)_{i \in A}$, we can define $F(v, \alpha)$ to be the set of all Nonsymmetric bargaining games for various choices of disagreement points $\mathbf{c}$ and set $T \subseteq A$.

We will assume that the following two natural conditions are satisfied by the function v :

1. **Non degenerate:** $v(\emptyset) = 0$.

2. **Non redundancy of sets:** For all subsets $S \subseteq A$, there exists a feasible utility vector $\mathbf{u}$ such that set S is *tight* w.r.t. $\mathbf{u}$, that is, $\sum_{i \in S} u_i = v(S)$. It is easy to see (using duality) that this property is equivalent to the property that v satisfies fractional covering property, that is, $\forall S$, $v(S) \leq \sum_{B \subseteq A} v(B) x_B$, where x_B's are such that $\forall i \in S: \sum_{B:i \in B} x_B \geq 1$.

We call such functions *valid* functions. Note that the second condition is without loss of generality as one can always modify the function v to satisfy this property without losing any of the feasible points. The second condition also implies (1) *Monotonicity:* for any $Z_1 \subset Z_2 \subseteq A$, we have $v(Z_1) \leq v(Z_2)$, and (2) *Complement freeness:* $v(Z_1 \cup Z_2) \leq v(Z_1) + v(Z_2)$.

2.2 Properties of UNB games

In this paper, we are interested in the following game theoretic properties of UNB games.

1. **Utilitarian Efficiency**: For any valid function $v : 2^A \to R^+$, we say that $F(v)$ is *Utilitarian Efficient* if for all subsets $T \subseteq A$, $\min_{(\mathbf{c},T) \in F(v)} u^*(c,T)$ is equal to $v(T)$, where $u^*(c,T) = \sum_i u_i^*(c,T)$.

2. **Fairness**: For any instance $I = (\mathbf{c}, T) \in F(v)$, define $core(I)$ to be the set of all feasible Pareto optimal solutions. For any vector $\mathbf{u}$, let $\mathbf{u}_{dec}$ be the vector obtained by sorting the components of $\mathbf{u}$ in decreasing order. A vector $\mathbf{x}$ *min-max dominates* $\mathbf{y}$ if $\mathbf{x}_{dec}$ is lexicographically smaller than $\mathbf{y}_{dec}$. Also let $\mathbf{u}^*$ be the bargaining solution of instance I. Instance I is said to be *min-max fair* if the vector $\mathbf{u}^* - \mathbf{c}$ *min-max dominates* $\mathbf{y} - \mathbf{c}$ for all $\mathbf{y} \in core(I)$. $F(v)$ is said to be *min-max fair* if all the instances in $F(v)$ are min-max fair. Similarly, we define the notion of *max-min fairness*.

3. **Strong monotonicity**: For any valid function $v : 2^A \to R^+$, we say that $F(v)$ is *strongly monotone* if, for all games in $F(v)$, the following property holds: On increasing the disagreement utility c_i of an agent i, the bargaining solution doesn't increase the utility for any other agent j, where $j \neq i$. Formally, for any instance $(\mathbf{c}, T)$ and $(\mathbf{c}', T)$, $T \subseteq A$, where $\mathbf{c}' = \mathbf{c}$ except that $c_i' > c_i$, then we have for all $j \neq i$, $u_j^*(\mathbf{c}', T) \leq u_j^*(\mathbf{c}, T)$.

4. **Surplus monotonicity**: For any valid function $v : 2^A \to R^+$, we say that $F(v)$ is *surplus monotone* if the surplus (i.e., $u_i^* - c_i$) of any agent i in the bargaining solution does not increase if her disagreement utility c_i increases. Formally, for any instance $(\mathbf{c}, T)$ and $(\mathbf{c}', T)$, $T \subseteq A$, where $\mathbf{c}' = \mathbf{c}$ except that $c_i' > c_i$, then we have $u_i^*(\mathbf{c}', T) - c_i' \leq u_i^*(\mathbf{c}, T) - c_i$.

5. **Population monotonicity**: For any valid function $v : 2^A \to R^+$, we say that $F(v)$ is *population monotone* if, for any $T_1 \subset T_2 \subseteq A$ and any agent $i \in T_1$, agent i cannot obtain more utility in the bargaining solution of instance $(\mathbf{c}, T_2)$ than in the bargaining solution of $(\mathbf{c}, T_1)$, i.e. $u_i^*(\mathbf{c}, T_1) \geq u_i^*(\mathbf{c}, T_2)$.

In this paper we show that for each of the above properties, other than surplus monotonicity, the property holds for $F(v)$ if and only if v is submodular.

All of the above definitions can be naturally extended for $F(v, \alpha)$ - the family of nonsymmetric bargaining games. Most of our results also extend naturally to this family of games; we point out the key differences in Section 10. For the case of *population monotonicity*, one can instead consider a stronger version, which says that bargaining utility of an agent shouldn't decrease as the clout of some other agent is decreased. We again show that $F(v, \alpha)$ satisfies population monotonicity w.r.t clouts if and only if v is submodular. From this point on till Section 10, we will concern ourselves with the Nash bargaining game.

3   Examples of UNB and SNB games

We give some natural examples of UNB and SNB games here. Example 1 illustrates a game which is in SNB; Example 2 illustrates a game which is in UNB but not in SNB; Example 3 illustrates a game which is in LNB but not in UNB. In each of these examples we only give the feasible set of utilities. The disagreement point $\mathbf{c}$ could be any point in the strict interior of the feasible set.

Example 1 *(Sharing arcs of a network with single-source)* Consider a directed network $N = (V, A)$ with capacities on arcs. Let there be a fixed source s in the network. Suppose there is a set of n agents where each agent controls a node t_i and is interested in receiving flow from the source s. The utility of an agent i is the amount of flow that goes from source s to sink t_i. For any set S of agents, let $v(S)$ be the size of min cut separating source s and set S. Note that the function v is a submodular function [Meg74]. Now the feasible polytope of utilities is given by

$$\mathcal{P} := \{\mathbf{u} : \forall\ S \subseteq A\ \sum_{i \in S} u_i \leq v(S);\ \forall i\ u_i \geq 0\}.$$

Example 2 *(Branchings in a network)* As in the previous example, consider a directed graph $G = (V, A)$ with capacities on the arcs. Agents are subsets of V. For an agent $s \in V$, let her desired object be branchings rooted at s and spanning all V, i.e., directed trees rooted at s and containing a path from s to each vertex in V. Suppose agent s sends a flow f_{sb} to each vertex in V using the branching object b, then his total utility is equal to the total flow sent along all his branchings, which is equal to $\sum_b f_{sb}$. For $S \subseteq V$, let $v(S)$ be the capacity of the minimum cut separating a vertex in $V \setminus S$ from S. A result of [JV10] on the characterization of the feasible set of utilities shows that the game lies in (UNB\SNB), when the number of agents $|A| \geq 3$. Moreover, the feasible utility set is given via

$$\mathcal{P} := \{\mathbf{u} : \forall\ S \subseteq V\ \sum_{i \in S} u_i \leq v(S);\ \forall i\ u_i \geq 0\}.$$

Example 3 *(Sharing arcs of a network with source-sink pairs)* Consider a directed graph $G = (V, A)$ where the arcs have capacities and there are k source-sink pairs $(s_1, t_1) \ldots (s_k, t_k)$ each being a player in the game. The utility of each agent i is the flow f_i that can be routed from s_i to t_i concurrently. Since all feasible flows from the sources to the sinks can be written as an LP, this game is in LNB.

We give a small example with $k = 2$ to show that this game is not in UNB. There are two arc-disjoint directed paths from s_1 to t_1 each of length $3 - (s_1, a, b, t_1)$ and (s_1, x, y, t_1), where a, b, x, y are nodes in the network; and there is a single directed path from s_2 to $t_2 - (s_2, a, b, x, y, t_2)$. The only arcs that have capacities are (a, b) and (x, y) and each has a capacity of 1. Observe that when agent 2 sends δ units of flow from s_2 to t_2, she "eats up" δ units on *each* path of agent 1. Moreover, if agent 2 were absent, agent 1 could've sent a flow of 2 units (on each of her paths) while agent 2 could send a maximum of 1 unit of flow in the absence of agent 1. That is, the two agents do not share the resources in a uniform manner. The feasible polytope of utilities in this example is given by

$$\mathcal{P} := \{(u_1, u_2) : 0 \leq u_1 \leq 2,\ 0 \leq u_2 \leq 2;\ \ u_1 + 2u_2 \leq 2\}.$$

We remark that this game, for the case of 2 players, and in fact all 2-player games in LNB have rational solutions and can be solved in polynomial time. This follows from a result of [CDV10].

4 Preliminaries

For any valid function v, we say that S is *tight* w.r.t. $\mathbf{u}$ if $\sum_{i \in S} u_i = v(S)$. Let $\mathbf{u}^*$ be the solution to the convex program given in (1). KKT conditions of optimality of convex programs imply the existence of Lagrangean duals $p(S)$ for every set $S \subseteq T$ satisfying the following three conditions.

1. $\forall S \subseteq T$, $p(S) \geq 0$.

2. $\forall S \subseteq T$, $p(S) > 0 \Rightarrow \mathbf{u}^*$ makes set S tight.

3. $\forall k \in T$, we have $u_k^* > c_k \Rightarrow \sum_{S: k \in S} p(S) = \frac{1}{u_k^* - c_k}$.

We will call $p(S)$ to be the price of set S.

The following lemma shows that for any set $T \subseteq A$ of agents and any Pareto optimal point $\mathbf{u} \in \mathcal{P}$, there exists a disagreement vector $\mathbf{c}$ such that $\mathbf{u}$ is the bargaining solution for $(\mathbf{c}, T)$.

Lemma 4.1. *Given any valid function v, a set $T \subseteq A$ of agents, and a utility vector $\mathbf{u}$ with $u_i > 0$, $\forall i \in T$, $\mathbf{u}$ is Pareto optimal if and only if there exists a vector $\mathbf{c}$, with $c_i > 0$ $\forall i \in T$, such that $\mathbf{u}$ is the bargaining solution for the instance $(\mathbf{c}, T)$.*

Proof. If $\mathbf{u}$ is a bargaining solution it has to be Pareto optimal. We now prove the converse. Let $\delta := \min_{i \in T} u_i > 0$. If $\mathbf{u}$ is Pareto optimal, we cannot increase u_i without changing other coordinates of $\mathbf{u}$. Therefore, every agent $i \in T$ is in (at least) one tight set $Z_i \subseteq T$ w.r.t. $\mathbf{u}$. Set the price of any tight set $Z_i \subseteq T$ to be $p(Z_i) = P$, where $P > \frac{1}{\delta}$. Define c_i's as follows.

$$c_i = u_i - \frac{1}{\sum_{Z \subseteq T : i \in Z} p(Z)}. \tag{2}$$

We claim that the $\mathbf{u}, \mathbf{c}$ and the prices satisfy the KKT conditions implying $\mathbf{u}$ is the Nash bargaining solution with disagreement vector $\mathbf{c}$. This is because only tight sets are prices and c_i's are so defined to satisfy the second KKT condition. It remains to check $c_i > 0$ for all i. This is true because every agent in T is in at least one tight set, and therefore $\sum_{Z: i \in Z} p(Z) \geq P > 1/\delta$. This gives $c_i > u_i - \delta \geq 0$. $\square$

Now we give some properties of the submodular and non-submodular functions which will be used in our proofs.

Property 4.1. *Given a valid submodular function $v : 2^A \to R_+$, and a utility vector $\mathbf{u}$, if $Z_1, Z_2 \subseteq A$ are tight sets w.r.t. $\mathbf{u}$, then $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are also tight sets w.r.t. $\mathbf{u}$.*

Proof. $u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2) = u(Z_1) + u(Z_2) = v(Z_1) + v(Z_2) \geq v(Z_1 \cup Z_2) + v(Z_1 \cap Z_2) \geq u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2)$, where the second equality follows from the tightness of Z_1 and Z_2. $\square$

By using the uncrossing argument and the above property, we get the following corollary.

Corollary 4.1. *Given any valid submodular function v, and $(\mathbf{c}, T) \in F(v)$, we can choose the prices for all subsets of T in the KKT conditions, such that the tight sets with positive prices form a nested set family, i.e. $T = T_1 \supset T_2 \supset \cdots T_k \supset T_{k+1} = \emptyset$.*

We next state a property of non-submodular valid functions which enhances the following theorem of [CD09].

Theorem 4.1. *(Theorem 3.1 of [CD09]) Given any valid non-submodular function v, there exists a set $S \subset A$, $i, j \in A \setminus S$, and a feasible utility vector $\mathbf{u}$ such that:*

1. *S, $S \cup \{i\}$, $S \cup \{j\}$ are all tight w.r.t $\mathbf{u}$.*

2. *No set containing both i and j is tight.*

3. *All tight sets containing either i or j, must contain the agent $\ell \in S$ with $u_\ell > 0$.*

[CD09] prove the above theorem by choosing the set S and $i, j \in A \setminus S$ to be the minimal set which violates submodularity, that is, $v(S \cup i \cup j) + v(S) > v(S \cup i) + v(S \cup j)$. Since S is minimal, the restriction of v to the set $S \cup i$ (or $S \cup j$) is indeed submodular. Since v is valid, there is at least one utility vector $\mathbf{u}$ which makes S tight. Of these, $\mathbf{u}$ is chosen which minimizes the number of tight subsets of S. u_i and u_j are defined so that $S \cup i$ and $S \cup j$ are tight, establishing part 1. Non-submodularity implies that the set $S \cup i \cup j$ is non-tight. [CD09] show that this is indeed true for any set containing both i and j establishing part 2. Establishing part 3 requires more work; [CD09] show that if there exist tight sets containing i and j which are disjoint, then one can show that the set $S \cup i \cup j$ is also tight, which is not possible. We refer the interested reader to their paper for full details. We now extend the above theorem in a straightforward way.

Property 4.2. *Given any valid non-submodular function v, there exists a set $S \subset A$, $i, j \in A \setminus S$, $\ell \in S$, and a feasible utility vector $\mathbf{u}'$ such that:*

(a) *v is submodular on $S \cup \{i\}$.*

(b) *$S \cup \{i\}$, $S \cup \{j\}$ are both tight.*

(c) *Let $\mathcal{F}_k$ denote the set of all subsets which contain k and are tight w.r.t $\mathbf{u}'$. We have*

$$\mathcal{F}_\ell = \mathcal{F}_i \cup \mathcal{F}_j, \quad \mathcal{F}_i \cap \mathcal{F}_j = \emptyset.$$

(d) *$u_k > 0$, $\forall k \in S \cup i \cup j$.*

Proof. Let $S, i, j, \mathbf{u}$ be as in Theorem 4.1. As sketched in the proof above, the set S and $i, j \in T \setminus S$ are chosen so that S is the minimal set contradicting submodularity. That is, v restricted to $S \cup i$ is submodular. Note that conditions (a) and (b) are satisfied. In part (c), we have $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ from part 2 of the above theorem. Part 3 implies $\mathcal{F}_i \cup \mathcal{F}_j \subseteq \mathcal{F}_\ell$; we need equality. Furthermore, part (d) might not be true since u_k could be 0 for some k.

We now modify the utility vector $\mathbf{u}$ to another vector $\mathbf{u}'$ such that $\mathbf{u}'$ satisfies our requirements. First of all, we show that we may assume without loss of generality that $u_k > 0$ for all $k \in S$. If there is any $r \in S$ such that $u_r = 0$, we remove the agent r from set S: let $S' = S \setminus \{r\}$. Note the element ℓ in condition 3 of the above theorem remains in S'. Note that $S', i, j, \ell, \mathbf{u}$ still satisfy the conditions in Theorem 4.1. $S', S' \cup i, S' \cup j$ are tight since the total utility of these sets remain the same, and since the larger sets were tight, the valuations remain the same due to feasibility of $\mathbf{u}$ and monotonicity of v. Part 2 and 3 are easy to check. Thus, $u_k > 0$ for all $k \in S$. Also note that the validity of part (a) is retained since S' is a subset of S.

9

Now define $\mathbf{u}'$ as follows.

$$u'_i = u_i + \epsilon, \ u'_j = u_j + \epsilon, \ \text{and } u'_\ell = u_\ell - \epsilon, \quad u'_k = u_k \text{ for all } k \in T \setminus \{i, j, \ell\}$$

where ϵ is defined as follows to keep $\mathbf{u}'$ feasible.

$$\epsilon < \min\{\epsilon_0, u_\ell/2\}, \ \text{where } \epsilon_0 := \min_{\text{nontight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

Since T is not tight, ϵ_0 is well defined and strictly positive. First we show that $\mathbf{u}'$ is feasible. For any set $Z \subseteq T$ which is not tight w.r.t $\mathbf{u}$, we have

$$\sum_{k \in Z} u'_k < \sum_{k \in Z} u_k + 2\epsilon_0 \leq v(Z).$$

Thus a non-tight set remains non-tight.

For any tight set $Z \subseteq T$ w.r.t $\mathbf{u}$, since we know that Z doesn't contain both i and j, there are two cases. If Z contains either i or j, then it contains l as well. Thus the total utility of that set doesn't change. Furthermore, the set remains tight. If Z doesn't contain i or j, then its total utility decreases implying feasibility.

We end by checking all conditions (a),(b),(c) and (d) are satisfied. Part (a) is not affected by definition of $\mathbf{u}'$ and is satisfied. Since tight sets containing i or j remain tight, part (b) remains true. Since non-tight sets remain non-tight, we have $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$. Since tight sets containing i and j remain tight, we have $\mathcal{F}_i \cup \mathcal{F}_j \subseteq \mathcal{F}_\ell$. Finally, any tight set containing ℓ and not i or j becomes non-tight. So, we have equality in the above subset relation. Part (d) is true since $\mathbf{u}'_i, \mathbf{u}'_j > 0$ and for any $k \in S$, we have $\mathbf{u}'_k > 0$ as well. $\qquad\square$

5 Efficient Algorithms

We will show that there is a combinatorial, strongly polynomial algorithm for solving each game in SNB; as a corollary we'll get that all such games are rational. Our result and algorithm generalizes that of [JV10], giving a combinatorial, strongly polynomial algorithm for finding the equilibrium for any Eisenberg-Gale market in the class SUA (submodular utility allocation markets).

Recall a set S is tight w.r.t $\mathbf{u}$ if $\sum_{i \in S} u_i = v(S)$. An agent $i \in T$ is called *free* if it exists in no tight set. If v is submodular, then by Property 4.1, if no agent in T is free, then T is tight. In particular, the set of all non-free agents is tight.

We will repeatedly use a strongly polynomial time, combinatorial algorithm to minimize submodular functions. Such an algorithm is due to Iwata et al. [IFF01] and Schrijver [Sch00] for general submodular functions. In particular, this gives us a strongly polynomial time, combinatorial subroutine to check whether a particular utility allocation $\mathbf{u}$ is feasible or not – it amounts to checking if the minimum value of the submodular function $v'(S) := v(S) - \mathbf{u}(S)$ is at least 0 or not. Furthermore, given a feasible utility allocation, the above subroutine can be used to obtain a maximal tight set; a set S is tight iff $v'(S) = 0$, and, by repeatedly shrinking a tight set and moving to residual set systems, one can get the maximal tight set. This is standard, and we refer the reader to [Sch03] for more details.

The algorithm starts off by assigning each client in T initial utility $u_i = c_i$. If this initial allocation is infeasible or even if there exists a tight set, the algorithm returns the game is infeasible.

The algorithm maintains a collection $\mathcal{T}$ of tight sets, prices of tight sets, and allocation u_i for each agent, and terminates when the KKT conditions for (1) are satisfied.

Algorithm for SNB Games:

1. Feasibility Check. Initialize $u_i = c_i$ for all $i \in T$. If this allocation is infeasible or there is a tight set, return the game is infeasible. Initialize $\mathcal{T}$ to be empty. Set time $t = 0$.

2. While there exists a free agent $i \in T$

   - Utility Augmentation. Increase t and u_i for all the free agents i at rate 1, until some new set X gets tight.
   - Set Pricing. Pick X to be the unique, maximal (inclusion-wise) tight set and put it in $\mathcal{T}$. Set the price of X, at $p(X) = 1/t$. For all maximal subsets $Y \in \mathcal{T}$ of X, decrease the price $p(Y) = p(Y) - 1/t$. Since $p(Y) = 1/t'$ for some $t' < t$, $p(Y)$ remains positive, furthermore, it is never decreased again since it'll never be a maximal subset subsequently.

Theorem 5.1. *The above algorithm terminates, and returns either that the game is infeasible, or an optimal solution to* (1).

Proof. If the feasibility check passes, then the game is feasible. At each iteration, a new agent participates in a tight set. The algorithm terminates in at most $|T|$ iterations. We now show that the prices returned satisfies the KKT conditions for (1).

Non-negativity is clear from the discussion in the price setting step. Furthermore, only tight sets are priced. Finally, note that the price faced by an agent, $\sum_{S:i\in S} p(S)$, remains unchanged from the time she becomes tight. This is guaranteed by the decrease in the prices of the maximal subsets of a newly formed tight set. So, the price faced by an agent is precisely $1/t$, where t is the time she entered the tight set. Since the utility of the agent doesn't change henceforth, $u_i = t + c_i$, ensuring the third KKT condition. $\qquad\square$

We end this section noting two corollaries. Firstly, observe that if $v(S)$ is a rational number for all subsets S, then the optimum solution and prices are all rational numbers as well. Thus, SNB games always have rational solutions, with any number of agents. This is in contrast to UNB games which may have irrational solutions when the number of agents is at least 3 (similar to examples in [JV10]). Furthermore, any function v satisfying the covering property is submodular for $|A| = 2$. So, all two agent UNB games are in SNB and thus can be solved in strongly polynomial time.

6 Utilitarian Efficiency

Recall that $F(v)$ is utilitarian efficient if $u^*(c, T)$ is equal to $v(T)$, for all feasible $(\mathbf{c}, T) \in F(v)$. We prove the following theorem.

Theorem 6.1. *For any valid function v, $F(v)$ is utilitarian efficient iff v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. We want to show that for any disagreement point $\mathbf{c}$, and set $S \subseteq A$, if we restrict to the subproblem among agents in S, the Nash bargaining solution $\mathbf{u}^*$ satisfies $\sum_{i\in S} u_i^* = v(S)$. Since $\mathbf{u}^*$ is the solution of the Nash bargaining game, it must be Pareto optimal.

11

Therefore every agent i is in some tight set T_i. Therefore by Property 4.1, we have $S = \cup_{i \in S} T_i$ is also tight, which means $\sum_{i \in S} u_i^* = v(S)$.

$\Rightarrow$: Suppose v is not submodular. By Property 4.2, there is a set $T = S \cup \{i, j\}$ and a feasible utility vector $\mathbf{u} = (u_k)_{k \in T}$ such that: (1) $u_k > 0, \forall k \in T$, (2) $S \cup i$ and $S \cup j$ are tight w.r.t. $\mathbf{u}$, (3) T is not tight w.r.t. $\mathbf{u}$. This is obtained from $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

Now for any $k \in T$, k is in some tight set w.r.t $\mathbf{u}$, hence by the lemma 4.1 in Section 4, there exists $\mathbf{c}$ such that $\mathbf{u}$ is the Nash bargaining solution corresponding to $\mathbf{c}$. By condition 3 above, we have $\sum_{k \in T} u_k < v(T)$, which implies that it is not utilitarian efficient. $\qquad \square$

7 Fairness

In this section, we prove the following theorem.

Theorem 7.1. *For any valid function v, $F(v)$ is min-max and max-min fair if and only if v is submodular.*

Proof. $\Leftarrow$: Suppose v is submodular. Let $\mathbf{u}^*$ be the Nash bargaining solution for $(\mathbf{c}, T)$ where $T \subseteq A$. By corollary 4.1, we can choose the prices such that the tight sets w.r.t $\mathbf{u}^*$ with positive price form a nested set family, $T = T_1 \supset T_2 \supset ...... \supset T_t \supset \emptyset$. Since for any agent i, $u_i^* - c_i = 1 / \sum_{j:i \in T_j} p(T_j)$, we see that $(\mathbf{u}^* - \mathbf{c})_{dec}$ has elements of $T_1 \setminus T_2$ followed by those in $T_2 \setminus T_3$ and so on. Moreover, agents in $T_j \setminus T_{j+1}$ have the same $u_i^* - c_i$.

Pick any element $\mathbf{g} \neq \mathbf{u}^*$ in $core((\mathbf{c}, T))$. Suppose $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$. Since $\mathbf{g}$ is Pareto optimal, every agent is in some tight set w.r.t $\mathbf{g}$. Hence by property 4.1, the whole set T is tight, $\sum_{k \in T} g_k = v(T)$. Since $\mathbf{g}$ is feasible, we also have

$$\sum_{k \in T_2} g_k \leq v(T_2).$$

Since T and T_2 are tight sets w.r.t $\mathbf{u}^*$, taking differences we get

$$\sum_{k \in T \setminus T_2} g_k \geq \sum_{k \in T \setminus T_2} u_k^*. \tag{3}$$

Since each agent i in $T \setminus T_2$ has the highest $u_i^* - c_i$ among all the agents, if $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$, then for any $k \in T \setminus T_2$, we have $g_k \leq u_k^*$. Then by (3), we have $g_k = u_k^*, \forall k \in T \setminus T_2$. Similarly, we can show for any $1 \leq i \leq t$ and any $k \in T_i \setminus T_{i+1}$, $g_k = u_k^*$. Hence $\mathbf{g} = \mathbf{u}^*$ which is a contradiction. This proof also shows that $\mathbf{u}^* - \mathbf{c}$ is the unique min-max fair utility vector. By using an argument similar to [JV08], we can show that any unique min-max fair utility vector is also max-min fair.

$\Rightarrow$: Suppose v is not submodular, then by property 4.2, there is a set $T = S \cup \{i, j\}$ and a $\mathbf{g} = (g_k)_{k \in T}$ such that: (1) $g_k > 0, \forall k \in T$, (2) $S \cup \{i\}$ and $S \cup \{j\}$ are tight w.r.t $\mathbf{g}$, (3) $\mathcal{F}_\ell = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$. For each $k \in T$, let $c_k = g_k - \epsilon$, where $0 < \epsilon < \min_{k \in T} \{g_k\}$. Clearly $\mathbf{g}$ is a feasible core element corresponding to $\mathbf{c}$, since each k is in a tight set (either $S \cup \{i\}$ or $S \cup \{j\}$).

Let $\mathbf{u}^*$ be the Nash bargaining solution corresponding to $(\mathbf{c}, T)$. Note that by definition, $\mathbf{g} - \mathbf{c}$ is a vector with all entries equal to ϵ. Thus, since $\mathbf{g}$ is Pareto optimal, it is the unique min-max and max-min feasible solution. We now show that $\mathbf{g}$ cannot be the solution to the Nash bargaining game implying the solution, $\mathbf{u}^*$ is not fair.

If **g** were the solution, by KKT conditions, we can price all the subsets of T such that:

$$\frac{1}{g_\ell - c_\ell} = \sum_{Z \in \mathcal{F}_\ell} p(Z) = \sum_{Z \in \mathcal{F}_i} p(Z) + \sum_{Z \in \mathcal{F}_j} p(Z) = \frac{1}{g_i - c_i} + \frac{1}{g_j - c_j}$$

which contradicts the fact that $g_\ell - c_\ell = g_i - c_i = g_j - c_j = \epsilon$. $\qquad\square$

8   Strong Monotonicity and Surplus Monotonicity

In this section we show that any SNB game $F(v)$ is both strongly monotone and surplus monotone. Moreover, we also show that any UNB game which is strongly monotone must be an SNB game. We will be using the algorithm for computing the optimal solution and the dual KKT prices for (1), described in Section 5.

Theorem 8.1. *For any submodular valid function v, $F(v)$ is surplus and strongly monotone.*

Proof. Suppose the disagreement of some agent (say agent i) goes from c_i to $c_i + \delta$. Denote the new disagreement vector by $\mathbf{c}'$. Let $\mathbf{u}$ and $\mathbf{u}'$ be the bargaining utility vectors with disagreement points $\mathbf{c}$ and $\mathbf{c}'$ respectively. We will show that $\forall j, u'_j - c'_j \le u_j - c_j$. It is easy to see that this implies both surplus monotonicity and strong monotonicity.

   We will use the continuous time algorithm presented in Section 5 to show that $u'_j - c'_j \le u_j - c_j$, $\forall j$. We will use *algorithm C* and *algorithm C'* to denote the run of the alogrithm with disagreement vectors $\mathbf{c}$ and $\mathbf{c}'$ respectively.

   Now suppose that an agent j becomes non-free at time t in the algorithm C, i.e., $u_j - c_j = t$. We will show that by time t some set containing j will go tight in the algorithm C' as well, which will imply that $u'_j - c'_j \le t$.

   Let A be the minimal set containing j which becomes tight at time t in the algorithm C. Let $Q := \{k \in A \mid \text{agent k becomes tight in the algorithm C' before time t}\}$. If $j \in Q$, we are done. Thus assume that $j \notin Q$. Now the fact that agents in Q become tight before time t in the algorithm C' but not in the algorithm C implies that there exists a set Z_k containing i and $k \in Q$ that becomes tight in the algorithm C' before time t. Let $Z = \cup Z_k$. Since for submodular functions, union of tight sets is tight, we get that Z is tight in algorithm C' before time t.

   Now we want to show that the total utility accumulated by the set $Z \cup A$ in algorithm C' by time t is at least $v(A \cup Z)$. This is so because

$$u'(Z\cup A) = u'(A\backslash Q)+u'(Z) \ge u(A\backslash Q)+v(Z) = u(A)-u(Q)+v(Z) \ge v(A)-v(Q)+v(Z) \ge v(A\cup Z)$$

   The first inequality follows from definition of Q and the fact that $u'(Z) = v(Z)$, the second inequality from the tightness of A under u and feasibility of u, and the last inequality follows from submodularity of v.

   This implies that all the agents in the set $Z \cup A$ are tight by time t in algorithm C'. Hence we get that $u'_j - c'_j \le t$.

$\qquad\square$

Theorem 8.2. *If a valid function v is not submodular, then $F(v)$ is not strongly monotone.*

Proof. Since v is not submodular, by property 4.2 there must exist a set S and agents $i, j \in A \setminus S$, $l \in S$, and a feasible utility vector $\mathbf{u}$ such that: (1) $S \cup \{i\}$, $S \cup \{j\}$ are both tight w.r.t. $\mathbf{u}$, (2)$\mathcal{F}_\ell = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$, (3) $u_k > 0, \forall k \in T$, where $T = S \cup \{i, j\}$.

We will now construct an instance $(\mathbf{c}, T) \in F(v)$ which is not strongly monotone. Let $\delta = \min_{k \in T} u_k > 0$. For tight sets $S \cup \{i\}$, $S \cup \{j\}$, we set their prices to be $p(S, i)$, $p(S, j)$ respectively, where $p(S, i) = p(S, j) = P = \frac{2}{\delta}$. For any other set $Z \subseteq T$, we set its price $p(Z)$ to be zero.

Let

$$\forall k \in T \quad c_k = u_k - \frac{1}{\sum_{Z \subseteq T, k \in Z} p(Z)}$$

Since $S \cup \{i\}$ and $S \cup \{j\}$ are both tight, so for any $k \in T$, there exist at least one $Z \subseteq S$ such that $p(Z) = P$, and we have

$$c_k \geq u_k - \frac{\delta}{2} > 0.$$

By the definition of $\mathbf{c}$, all the KKT conditions hold, thus $\mathbf{u}$ is the bargaining solution w.r.t. $(\mathbf{c}, T)$.

Suppose there exists a $\mathbf{c}'$ and a corresponding bargaining solution $\mathbf{u}'$, such that: (1) $\forall k \in T, k \neq j, c'_k \geq c_k$, and (2) $c'_j = c_j$ and $u'_j > u_j$.

Using this, we can show that there exists a game in $F(v)$ which is not strongly monotone. This is because $\mathbf{c}'$ can be obtained from $\mathbf{c}$ by increasing only the coordinates other than j. If $F(v)$ is strongly monotone, then each time a coordinate of $\mathbf{c}$ is increased utility allocated to j shouldn't increase. But if $u'_j > u_j$ is true then we get a contradiction.

Let $\mathbf{u}'$ be same as $\mathbf{u}$ except that $u'_j = u_j + \epsilon$, $u'_i = u_i + \epsilon$, $u'_\ell = u_\ell - \epsilon$. Using arguments similar to the proof of property 4.2, one can show that there exists small enough ϵ (given below) so that $\mathbf{u}'$ is feasible.

$$\epsilon < \min\{\epsilon_0, u_\ell/2\}, \text{ where } \epsilon_0 := \min_{\text{non-tight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

Now we will construct $\mathbf{c}'$ satisfying the above mentioned conditions, and *price* the sets so that along with $\mathbf{u}'$ they satisfy the KKT conditions, which would imply that $\mathbf{u}'$ is the bargaining solution for the disagreement point $\mathbf{c}'$.

We assign positive price to sets $S \cup i$, $S \cup j$ only, say $p'(S, i)$ and $p'(S, j)$ respectively, which is consistent with second KKT condition as both the sets are tight under $\mathbf{u}'$. Third KKT condition says that: 1)$c'_i = u'_i - \frac{1}{p'(S,i)}$, 2)$c'_j = u'_j - \frac{1}{p'(S,j)}$, and 3)$c'_k = u'_k - \frac{1}{p'(S,i)+p'(S,j)}$, $\forall k \in T, k \neq i, j$. Thus to get $\mathbf{c}'$, one can equivalently find prices $p'(S, i)$ and $p'(S, j)$.

Now, since we want $c'_j = c_j$, we get that

$$u'_j - \frac{1}{p'(S, j)} = u_j + \epsilon - \frac{1}{p'(S, j)} \quad \Rightarrow \quad \epsilon - \frac{1}{p'(S, j)} = -\frac{1}{p(S, j)}$$

Similarly, for $k \neq j$, we want $c'_k \geq c_k$. Expanding c'_k and c_k for different k, we get that following necessary condition should hold

$$-\epsilon - \frac{1}{p'(S, i) + p'(S, j)} \geq -\frac{1}{p(S, i) + p(S, j)}$$

14

It is not difficult so see that one can find $p'(S, i)$ and $p'(S, j)$ which satisfies above two conditions as long as the ϵ is choosen such that $\epsilon < \frac{1}{p(S,i)+p(S,j)} = \frac{\delta}{4}$.

To sum up, by setting $\epsilon = \min\{\epsilon_0/2, \delta/8\}$, we can find $p'(S, i), p'(S, j)$ such that:

$$p'(S, j) = \frac{1}{\epsilon + \frac{1}{p(S,j)}}, \ p'(S, i) \geq \frac{1}{\frac{1}{p(S,i)+p(S,j)} - \epsilon} - p'(S, j)$$

Note that this value of ϵ is consistent with the previous mentioned upper bound on it. Therefore, we can construct $\mathbf{c}'$ such that $\mathbf{u}'$ is the bargaining solution w.r.t. $\mathbf{c}'$ and $c'_k \geq c_k, \forall k \in T, c'_j = c_j$. Thus $(\mathbf{c}, T) \in F(v)$ is not strongly monotone. $\square$

However, SNB games are not the only UNB games which are surplus monotone. The following is an example of a UNB game which is not an SNB game but is still surplus monotone. It is an interesting open question to characterize surplus monotone games based on their valuation functions.

Example 4. Consider the following game with three agents, $A = \{1, 2, 3\}$, and $v : 2^A \to R^+$ is defined as: $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 3$, $v(\{1,2\}) = v(\{2,3\}) = v(\{3,1\}) = 4$, $v(\{1,2,3\}) = 6$. This game is not an SNB game. However rate monotonicity holds. To prove this rigorously, one needs to do a case analysis. We sketch how this is done. It is not hard to see any bargaining solution either has $u_1 = u_2 = u_3 = 2$, or there is an agent, say agent 1, with utility between 2 and 3. In the former case, the sets $\{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}$ are tight and can have positive price. In the latter case, however, all tight sets contain the agent 1. Now suppose the disagreement of some agent j is increased and for contradiction's sake, assume that the difference $(u_j - c_j)$ also increases. Note, by the KKT conditions this is equivalent to saying that the price "faced" by agent j ($\sum_{S:j \in S} p(S)$) decreases. This cannot happen, and one can argue by going over all cases. We do one such case. Suppose agent j had utility between 2 and 3 (and thus all tight sets contain j). When c_j is increased and rate monotonicity is violated, u_j also increases. Thus, u_k decreases for all $k \neq j$ (since the only tight set containing $k \neq j$ contains j). Since c_k is the same, the price faced by k increases. Thus, with the new disagreements, the total price faced by $k \neq j$ increases, but total price faced by j decreases; this is not possible since all tight sets (priced sets) contain j. One can do the other cases similarly.

9 Population Monotonicity

In this section, we investigate population monotonicity in a UNB game. Generally speaking, population monotonicity means that when bargaining with a superset of agents, one cannot obtain more utility. This seems to be a reasonable property for bargaining games, however, it turns out that the population monotonicity only holds for SNB games.

Theorem 9.1. *For any valid function v, $F(v)$ is population monotone iff v is submodular.*

Proof. $\Leftarrow$: The proof of this is similar to the proof that SNB games are strongly monotone. Once again, we use the continuous time algorithm of Section 5 to prove that for submodular v, $F(v)$ is population monotone and urge the readers to recall the same.

Consider the run of the algorithm when the set of agents is T_1 and T_2. Let the utility vectors obtained be $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ respectively and let $\mathbf{y}^{(1)} := \mathbf{u}^{(1)} - \mathbf{c}(T_1)$ and $\mathbf{y}^{(2)} := \mathbf{u}^{(2)} - \mathbf{c}(T_2)$. It is enough to show that for all agents in T_1, $y_i^{(1)} \geq y_i^{(2)}$. Pick an agent $i \in T_1$ and suppose in the run

of the algorithm with T_1, at time t, i becomes non-free – that is, $y_i^{(1)} = t$. We now show that in the run of the algorithm with T_2, there exists a subset of T_2 containing i which is tight by time t (or over tight, which would imply $y_i^{(2)} < t = y_i^{(1)}$).

Let A be the first set containing i which goes tight in the run of the algorithm with T_1. Let $Q \subseteq A := \{j \in A : y_j^{(2)} < y_j^{(1)}\}$. We may assume that $i \notin Q$ for otherwise we are done. Moreover, for any $j \in Q$, $y_j^{(2)} < y_j^{(1)}$ implies that there must be a set $Z_j \subseteq T_2$ which goes tight in the run of the algorithm with T_2 and Z_j is not a subset of T_1. Let Z be the union of all these sets $\{Z_j : j \in Q\}$. Note that $Q = Z \cap A$ and that Z is tight.

Consider the set $Z \cup A$ – we have

$$y^{(2)}(Z \cup A) = y^{(2)}(Z) + y^{(2)}(A \setminus Q) \geq v(Z) + y^{(1)}(A \setminus Q)$$
$$= v(Z) + y^{(1)}(A) - y^{(1)}(Q) \geq v(Z) + v(A) - v(Q) \geq v(Z \cup A)$$

where the last but one inequality follows from the fact that A was tight w.r.t. $\mathbf{y}^{(1)}$, and $\mathbf{y}^{(1)}$ was feasible, and the last follows from submodularity of v.

$\Rightarrow$: If v is not submodular, recall that by property 4.2, there exists a set $S \subseteq A$, $\ell \in S$, $i, j \in A \setminus S$ and a feasible utility vector $\mathbf{u}$ such that the following hold:

1. $S \cup \{i\}$ and $S \cup \{j\}$ are both tight w.r.t $\mathbf{u}$.

2. $\mathcal{F}_\ell = \mathcal{F}_i \cup \mathcal{F}_j$ and $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

3. v is submodular on $S \cup \{i\}$.

4. for any $k \in T$, $u_k > 0$, where $T = S \cup \{i, j\}$.

For this utility $\mathbf{u}$, we can find $\mathbf{c}$ such that $\mathbf{u}$ is the bargaining solution of instance $(\mathbf{c}, T)$, i.e. $\mathbf{u} = \mathbf{u}^*(\mathbf{c}, T)$. We will show that after we remove agent j, there must be some agent in $S \cup \{i\}$ such that her utility in the bargaining solution of instance $(\mathbf{c}, S \cup \{i\})$ will be smaller.

Consider the instance $(\mathbf{c}, S \cup \{i\}) \in F(v)$, let the bargaining solution be $\mathbf{u}' = \mathbf{u}^*(\mathbf{c}, S \cup \{i\})$. Since v is submodular on $S \cup \{i\}$, we know that $S \cup \{i\}$ is tight w.r.t $\mathbf{u}'$. Recall that $S \cup \{i\}$ is also tight w.r.t $\mathbf{u}$, we have:

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

If no agent's utility differs in two bargaining solutions, the set of tight sets does not change. Consider $r_\ell := \frac{1}{u_\ell - c_\ell}$, in the solution of instance $(\mathbf{c}, T)$:

$$r_\ell = \sum_{Z \in \mathcal{F}_\ell} p(Z) = \sum_{Z \in \mathcal{F}_i} p(Z) + \sum_{Z \in \mathcal{F}_j} p(Z) = r_i + r_j$$

After we remove the agent j, since $\mathcal{F}_j$ is removed from $\mathcal{F}_\ell$, i.e. $\mathcal{F}'_\ell = \mathcal{F}'_i$, thus we have $r_\ell = r_i$, which is a contradiction. So at least one agent's utility differs. Since

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

there must be an agent $k \in S \cup \{i\}$ with $u'_k < u_k$. Therefore, if any UNB game $F(v)$ is population monotone, v must be submodular. $\qquad\square$

10 Extension to Nonsymmetric Nash Bargaining Games

Recall that $F(v, \alpha)$ is a family of nonsymmetric bargaining games, where function $v : 2^A \to R^+$ encodes the feasible set and $\alpha = (\alpha_i)_{i \in A}$ specifies the bargaining weights or clouts of all the agents.

Theorem 10.1. *For any valid function v and positive vector α, we have the following:*

1. *There is a strongly polynomial time, combinatorial algorithm to compute the solution of $F(v, \alpha)$ if v is submodular.*

2. *$F(v, \alpha)$ is utilitarian efficient iff v is submodular.*

3. *$F(v, \alpha)$ is max-min fair and min-max fair iff v is submodular.*

4. *$F(v, \alpha)$ is strongly monotone iff v is submodular.*

5. *$F(v, \alpha)$ is surplus monotone.*

6. *$F(v, \alpha)$ is population monotone w.r.t clouts iff v is submodular*

Proof. All the proofs can be easily extended; we give only key modifications that are required for some of the above properties:

(1) In the algorithm described in Section 5, instead of increasing u_i at rate 1 for all free agents, we raise u_i at rate α_i. The prices remain the same. The total price faced by an agent i in the end is precisely $\alpha_i/(u_i^* - c_i)$, satisfying the KKT condition of the new convex program. With this modification, the proof of (4) and (5) are the same as in the Nash case.

(2) In equation (2), which is based on the KKT conditions, we do the following modification:

$$c_i = u_i - \frac{\alpha_i}{\sum_{Z \subseteq T : i \in Z} p(Z)},$$

so that the lemma 4.1 still holds for the nonsymmetric case. Rest of the proof follows in similar fashion.

(3) For the case of fairness, we need to redefine the notion of fairness taking into account the relative bargaining powers of different agents. A bargaining game is said to be *min-max* (or *max-min*) fair if the vector $\{\frac{u_i^* - c_i}{\alpha_i}\}_i$ dominates $\{\frac{x_i - c_i}{\alpha_i}\}_i$ for all vectors $\mathbf{x}$ in the core. Proof follows on the same lines as the one given in section 7

(6) The proof for one direction, namely $F(v, \alpha)$ is *population monotone w.r.t. clouts* when v is submodular, follows from the algorithm and the arguments used in section 8. For the other direction, we use the same examples as given in [CD09] and keep the disagreement vector to be the $\mathbf{0}$ vector. □

11 Discussion

Many of our criteria, e.g., for efficiency or fairness, are the most stringent possible, i.e., characterizing games having *full* efficiency or max-min and min-max fairness. There is probably much to be gained by considering relaxed notions of these properties.

A more specific problem is that we have proved surplus monotonicity for games in SNB and have given examples to show that this does not characterize SNB within UNB. We leave the open problem of characterizing the family of games within UNB that are surplus monotonic.

We note that the study of these properties is not applicable to Kalai-Samaridinko (KS) bargaining games [KS75], since the KS solution is not well defined for UNB games when the number of players is more than two [Rot79] [2].

12   Acknowledgement

References

[Aum94] Cooperative models of bargaining. In R. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, pages 1237–1284, 1994.

[CD09] D. Chakrabarty and N. Devanur. On competitiveness in uniform utility allocation markets. In *Operations Research Letters*, pages 155–158, 2009.

[CDV10] D. Chakrabarty, N. Devanur, and V. V. Vazirani. Rationality and Strongly Polynomial Solvability of Eisenberg-Gale Markets with Two Agents. *SIAM J. of Discrete Math*, 24(3):1117–1136, 2010.

[IFF01] S. Iwata, L. Fleischer, and S. Fujishige. A strongly polynomial-time algorithm for minimizing submodular functions. *Journal of the ACM*, 48:761–777, 2001.

[JV08] K. Jain and V. V. Vazirani. Equitable cost allocations via primal-dual-type algorithms. *SIAM Journal on Computing*, 38(1):241–256, 2008.

[JV10] K. Jain and V. V. Vazirani. Eisenberg-Gale Markets: Algorithms and Game-Theoretic Properties. *Games and Economic Behavior*, 70(1):84–106, 2010.

[Kal77] E. Kalai. Nonsymmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory*, 6:129–133, 1977.

[Kal85] E. Kalai. Solutions to the bargaining problem. In L. Hurwicz, D. Schmeidler, and H. Sonnenschein, editors, *Social Goals and Social Organization*, pages 75–105. Cambridge University Press, 1985.

[KS75] E. Kalai and M. Smorodinsky. Other Solutions to Nash's Bargaining Problem. *Econometrica*, 43(3):513–518, 1975.

[Meg74] N. Megiddo. Optimal flows in networks with multiple sources and sinks. *Mathematical Programming*, 7:97–107, 1974.

[MS01] H. Moulin and S. Shenker. Strategyproof sharing of submodular costs: Budget balance versus efficiency. *Economic Theory*, 18(3):511 – 533, 2001.

[Nas50] J. F. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.

---

[2]Although Roth [Rot79] didn't explicitly mention that his impossibility example is in UNB, this is clear from the proof, and it is also not difficult to show it.

[OR90]   M. Osborne and A. Rubinstein. *Bargaining and Markets*. Academic Press, Inc., 1990.

[Rot79]   A. E. Roth. An impossibility result concerning n-person bargaining games. *International Journal of Game Theory*, 8(3):129–132, 1979.

[Sch00]   A. Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory, Series B*, 80:346–355, 2000.

[Sch03]   A. Schrijver. *Combinatorial Optimization*, volume B. Springer, 2003.

[Tho83]   W. Thomson. The fair division of a fixed supply among a growing population. *Mathematics of Operations Research*, 8(3):319–26, August 1983.

[Tho87]   W. Thomson. Monotonicity of bargaining solutions with respect to the disagreement point. *Journal of Economic Theory*, 42(1):50–58, June 1987.

[Tho94]   W. Thomson. Anonymity and population-monotonicity. *Mimeo*, 1994.

[Tho08]   W. Thomson. Bargaining and the theory of cooperative games: John Nash and Beyond. *Manuscript, To appear*, 2008.

[TL89]   W. Thomson and T. Lensberg. *Axiomatic Theory of Bargaining With a Variable Population*. Cambridge University Press, 1989.

[Vaz12]   V. V. Vazirani. The Notion of a Rational Convex Program, and an Algorithm for the Arrow-Debreu Nash Bargaining Game. *Journal of the ACM*, 59(2), 2012.

[vNM44]   J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.