Classical thermodynamics and economic general equilibrium theory

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Outline

- History and some conventions
- Modern neoclassical economics
- Structure of thermodynamics
- The right connection
- An example

A (very) little history

Parallel goals of "natural" and "social" physics circa 1900

- Define and characterize equilibria

Points of rest	"Best" resource allocations
Equations of state	Discovery of price systems

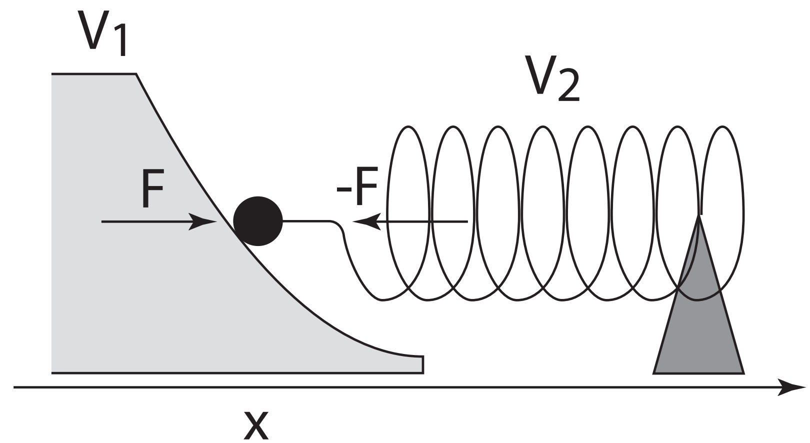
- Describe transformations

Work, heat flow	Trade, allocation processes

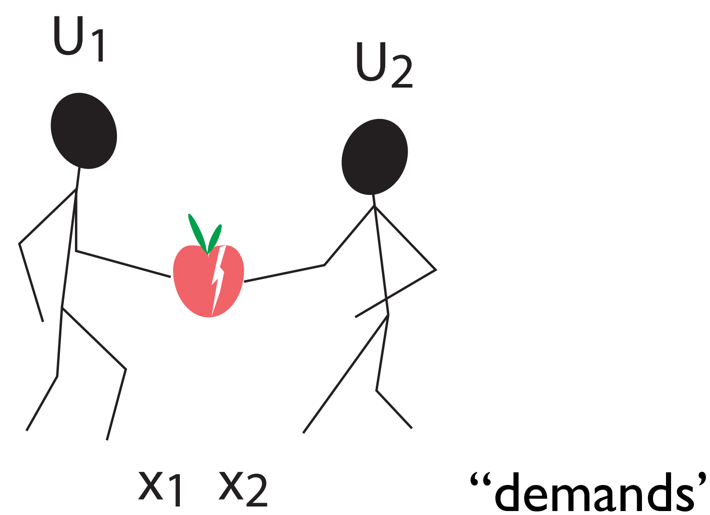
The Walrasian Analogy

Leon Walras (1909)

- Equilibrium as force balance in mechanics



- Equilibrium as balance of "marginal utility" in exchange



Analogies from mechanics

Position (x)	Holdings (x)
Potential Energy (V)	Utility (U)
Force (F)	Prices (p)
$$\mathrm{F} = -\nabla \mathrm{V}$$	$$\mathrm{p} = \nabla \mathrm{U}$$

(Utility is implicitly measurable)

~~“Ball settles in the bottom of the bowl to minimize energy”~~

Gibbs and thermodynamics

Distinction between *particle* and *system*

- Entropy is maximized in a closed system at equilibrium

$$S(U)$$

- For "open" subsystem, *excess* entropy is maximized

$$S(U) - \beta U$$

$$\beta = \frac{\partial S_{\mathrm{env}}}{\partial U_{\mathrm{env}}} \equiv \frac{1}{T}$$

- Helmholtz *Free Energy* is equivalently minimized

$$A = U - TS$$

Ball settles in the bottom of the bowl to maximize excess entropy (by losing energy)

Irving Fisher (1926)

And yet Fisher...

A particle	An individual
Space (x? V?)	Commodities (x)
Energy (U?, E?, V?)	Utility (U)
Force (F)	Marginal utility (p)

- Particles and individuals are unpredictable
- State variables are only properties of thermodynamic systems at equilibrium
- Fisher mixes metaphors from thermodynamics and statistical mechanics

Analogy and confusion

- J. H. C. Lisman (1949)

A quasi-eq. system	An individual
Entropy	Utility (“analogon”)
pV (ideal gas)	px (value)

- J. Bryant (1982)

$pV = NT$	$px = NT$ (productive content)

Disgust

The formal mathematical analogy between classical thermodynamics and mathematic economic systems has now been explored. This does not warrant the commonly met attempt to find more exact analogies of physical magnitudes -- such as entropy or energy -- in the economic realm. Why should there be laws like the first or second laws of thermodynamics holding in the economic realm? Why should ``utility" be literally identified with entropy, energy, or anything else? Why should a failure to make such a successful identification lead anyone to overlook or deny the mathematical isomorphism that does exist between minimum systems that arise in different disciplines?

Samuelson 1960

But duality survived

- Extensive quantities

Energy, volume — Goods

- Intensive quantities

Temperature, pressure — Prices

The marginalist revolution and modern "Neoclassical" mathematical economic theory

Indifference and utility

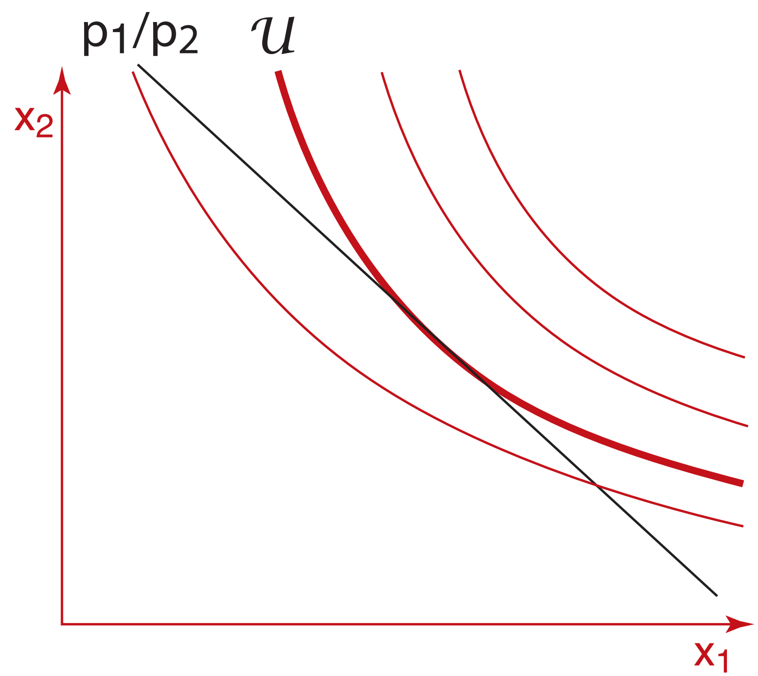
- Suppose more than one good

$$x = (x_0, x_1, \ldots, x_n)$$

- Only try to capture the notion of *indifference*

$$u(x) = \mathcal{U}$$

- Relative prices = marginal rates of substitution of goods
- "Absolute" price undefined



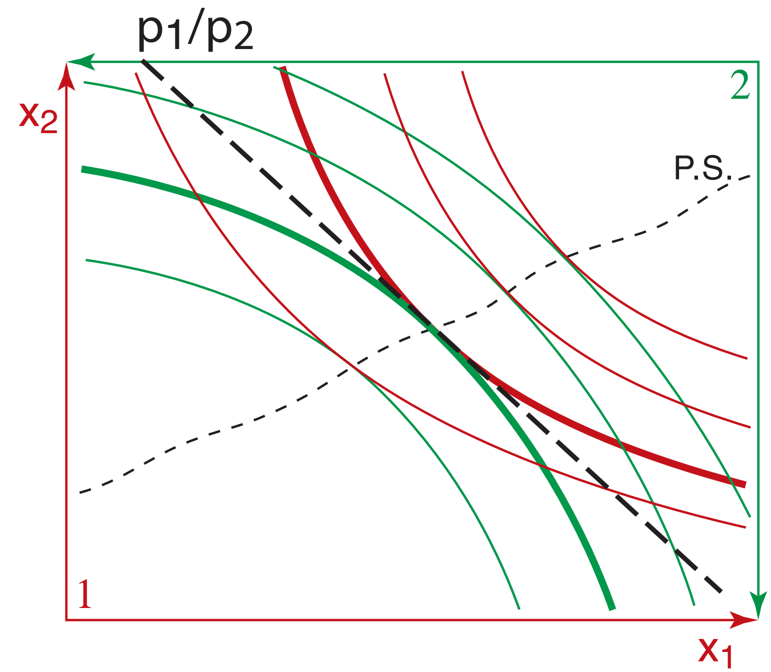
$$\frac{\partial u/\partial x_i}{\partial u/\partial x_i} = p_i/p_j$$

Utility is now explicitly only *ordinal*

The separating hyperplane

(Tj. Koopmans, 1957)

- "Edgeworth-Bowley" box: Conserve "endowments": (allocation of resources under conditions of scarcity)
- Prices separate agent decisions from each other (trade *and* production)
- "Pareto Optimum" defines equilibrium as no-trade
- Trade to equilibrium *must be irreversible*



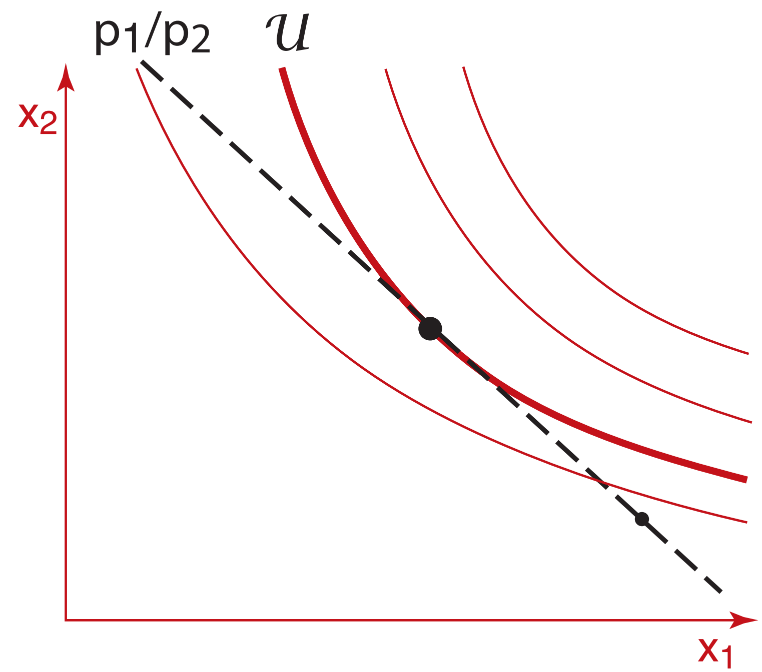
No trade any agent can propose from an equilibrium will be voluntarily accepted by any other agent

Duality: prices and demands

$$x = (x_0, x_1, \ldots, x_n)$$

$$u(x) = \mathcal{U}$$

$$\frac{\partial u}{\partial x_i} \propto p_i$$ "Offer prices"



$$e(p, \mathcal{U}) \equiv \min_x [p \cdot x \mid u[x] \geq \mathcal{U}]$$

Expenditure function

$$\delta e = \delta p \cdot x + p \cdot \left. \frac{\partial x}{\partial \mathcal{U}} \right|_p \delta \mathcal{U}$$

$$\left. \frac{\partial e}{\partial p_i} \right|_{\mathcal{U}} = x_i$$

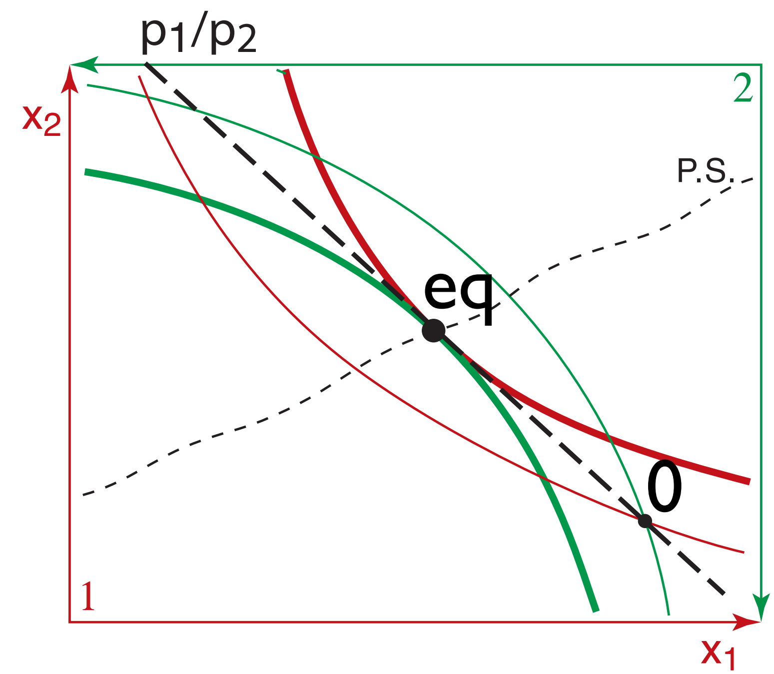
Exchange economies and the Walrasian equilibrium

$$x = (x_0, x_1, \ldots, x_n)$$

$$p = (p_0, p_1, \ldots, p_n)$$

Maximize:

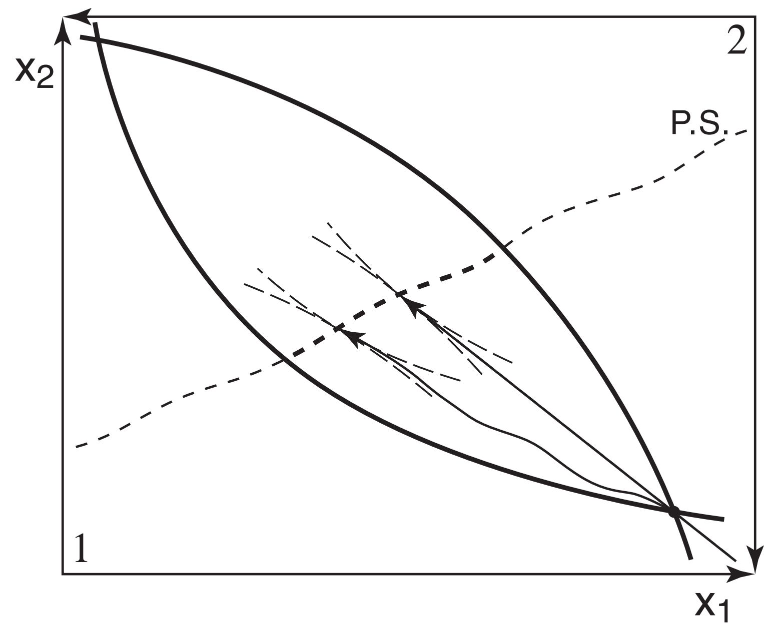
$$\mathcal{L} = u(x) - \beta p \cdot (x - x^0)$$



"Wealth preservation" hoped to extract a single equilibrium from the Pareto set

Trading paths to equilibrium *really aren't* determined

- The equilibrium price is a terminal property of real trade
- Need not restrict prior paths of trading
- The equilibrium price can be quite unrelated to the Walrasian price



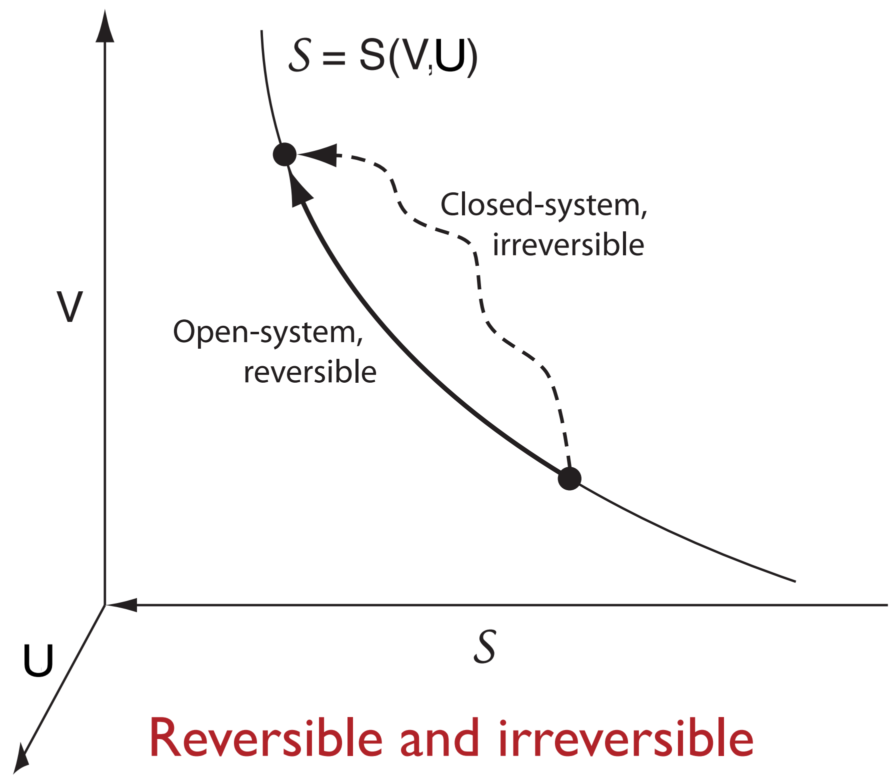
F. Hahn and T. Negishi (1962)

"and you may ask yourself 'how did I get here?' "

The mathematical structure of thermodynamics

State relations

- General statistical systems have E, $\mathcal{S}$, ***not*** predictable
- Only for equilibrium systems is E also a constraint U
- $S(V,U) = \max(\mathcal{S})|_{V,U}$ defines the "surface of state"
- Equation of state is not dependent on the path by which a point is reached

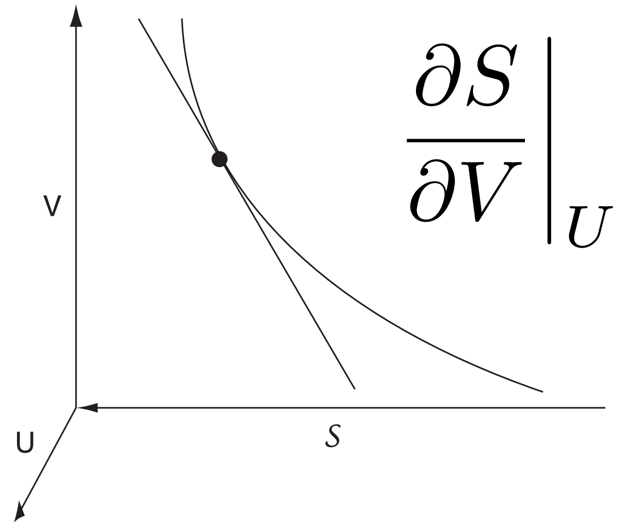


Reversible and irreversible transformations result in the same final state relation

Duality and Gibbs potential

State: $S(U,V)$

$$dS \equiv \frac{1}{T}dU + \frac{p}{T}dV$$



$$\left.\frac{\partial S}{\partial V}\right|_U = \frac{p}{T}$$

$$\delta\left(\frac{1}{T}U + \frac{p}{T}V - S\right) = U\,\delta\left(\frac{1}{T}\right) + V\,\delta\left(\frac{p}{T}\right)$$

$$\frac{G}{T} = \frac{U + pV - TS}{T} \qquad \left.\frac{\partial\,(G/T)}{\partial\,(p/T)}\right|_{1/T} = \left.\frac{\partial G}{\partial p}\right|_T = V$$

Connecting thermodynamics to mechanics

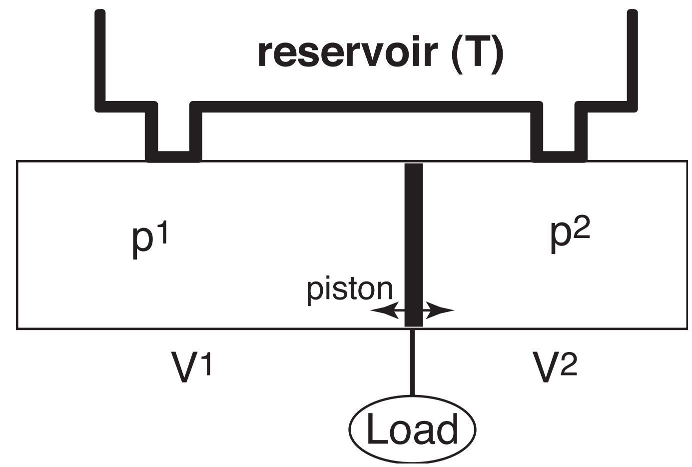
$$S(U,V) \qquad\qquad A(T,V) = U - TS$$

$$dS \equiv \frac{1}{T}dU + \frac{p}{T}dV \qquad\qquad dA = -pdV - SdT$$

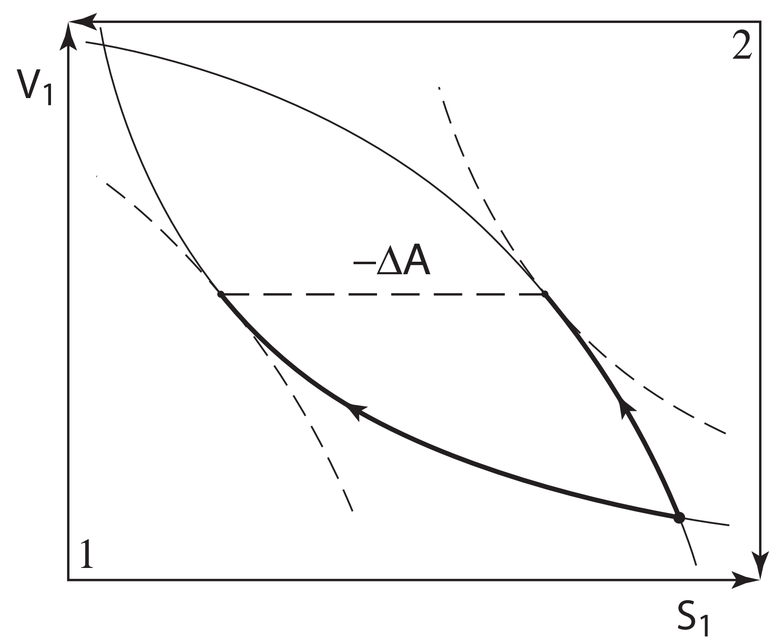
$$\left.\frac{\partial S}{\partial V}\right|_U = \frac{p}{T} \qquad\qquad -\left.\frac{\partial A}{\partial V}\right|_T = p$$

Reversible transformations and work

$$-\left.\frac{\partial A}{\partial V}\right|_T = p$$



$$\begin{aligned}\Delta W &= \int (p^1 - p^2)\, dV^1 \\ &= \int -\left(dA^1 + dA^2\right) \\ &= -\Delta A\end{aligned}$$



Helmholtz “free energy”

Analogies suggested by duality

Surface of state

$$S(V, U) = \max(\mathcal{S})|_{V,U}$$

Indifference surface

$$u(x) = \mathcal{U}$$

Increase of entropy

$$\delta S \geq 0$$

Increase of utility

$$\delta \mathcal{U} \geq 0$$

Intensive state variables

$$\left.\frac{\partial S}{\partial V}\right|_U = \frac{p}{T}$$

Offer prices

$$\frac{\partial u}{\partial x_i} \propto p_i$$

Gibbs potential

$$G = U + pV - TS$$

Expenditure function

$$e\left(p, \mathcal{U}\right) \equiv \min_x \left[p \cdot x \mid u\left[x\right] \geq \mathcal{U}\right]$$

Problems (1): counting

- Different numbers of intensive and extensive state variables (incomplete duality)

$$(U, V) \qquad\qquad x = (x_0, x_1, \ldots, x_n)$$

$$\left(\frac{1}{T}, \frac{p}{T}\right) \qquad\qquad \hat{p} \equiv (p_0, p_1, \ldots, p_n)\,/p_0$$

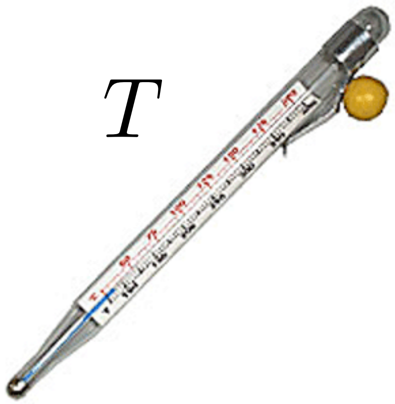
- Entropy is measurable, utility is not

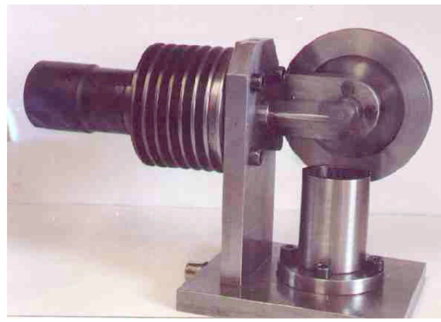
$$G(p, T) \qquad\qquad e(p, \mathcal{U})$$

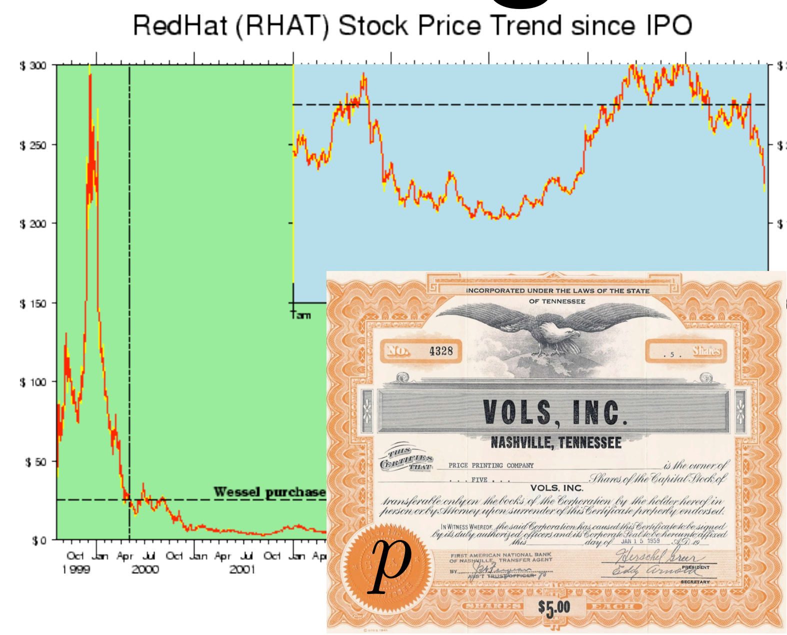
- *Total* entropy increases; *individual* utility does

$$\delta S \geq 0 \qquad\qquad \delta \mathcal{U} \geq 0$$

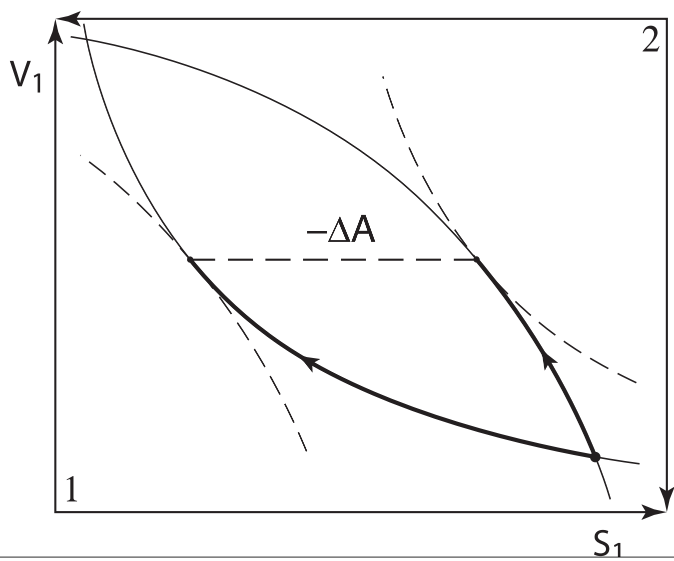
Problems (II): meaning

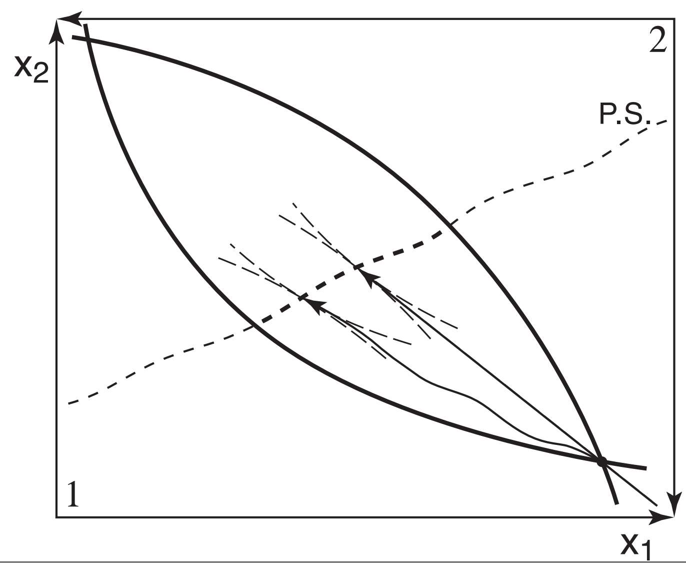






$$-pdV = dW = dU - TdS$$





Essence of the mismatch

- In physics, duality of *state* constrains transformations

The "price" of this power is that we must limit ourselves to reversible transformations, and cannot conserve all extensive state variable quantities

- In economics, conservation of endowments forces irreversible transformations

The result is that dual properties of state become irrelevant to analysis of transformations

Finding the right correspondence

Three laws in both systems

- Encapsulation

The state of a thermodynamic system at equilibrium is completely determined by a set of pairs of dual *state variables*

Economic agents are characterized by their holdings of *commodity bundles* and dual *offer price systems* to each bundle

- Constraint

Energy is conserved under arbitrary transformations of a closed system

Commodities are neither created nor destroyed by the process of exchange

- Preference

A partial order on states is defined by the *entropy*; transformations that decrease the entropy of a closed system do not occur

A partial order on commodity bundles is defined by utility; agents never voluntarily accept utility-decreasing trades

The construction

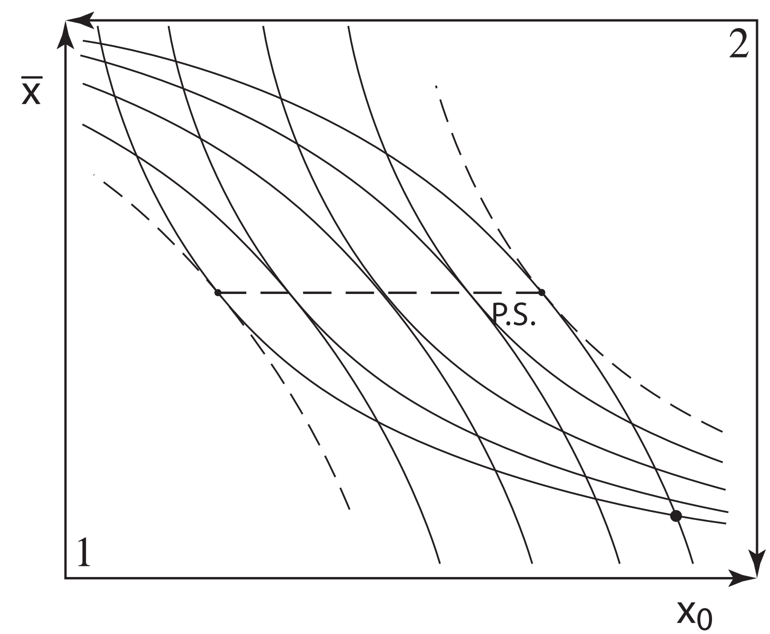
- Relate the surface of state to indifference surfaces correctly
- Study economics of reversible transformations
- Associate quantities by homology, not by analogy

Quasilinear economies: introduce an irrelevant good

$$x \equiv (x_0, \bar{x})$$

$$u(x) = x_0 + \bar{u}(\bar{x})$$

- Indifference surfaces are translations of a single surface in x_0 (hence so are all equilibria of an economy)
- All prices on the Pareto Set are equal
- Differences among equilibria have *no consequences* for future trading behavior



Duality on equivalence classes

$$\frac{\partial \bar{u}}{\partial x_i} = \frac{p_i}{p_0} \qquad \forall i > 0$$

Independent of distribution of x_0 among agents

Equivalence class of expenditures corresponds to Gibbs

$$e_{\mathrm{QL}}(p, \mathcal{U}) = p_0 \left[\mathcal{U} - \bar{u}\left(\bar{x}\right)\right] + \bar{p} \cdot \bar{x} \qquad p_0 \leftrightarrow T$$

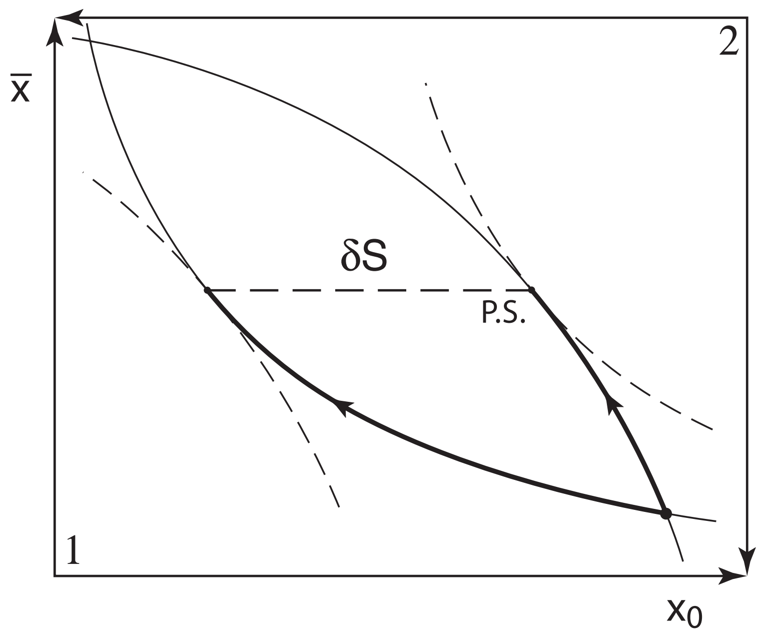
$$e_{\mathrm{QL}} - p_0 \mathcal{U} \leftrightarrow G = -TS + (U + pV)$$

Resulting economic entropy gradient is normalized prices

$$S_{\mathrm{QL}} = \bar{u}(\bar{x}) \qquad dS_{\mathrm{QL}} = d\bar{x} \cdot \frac{\bar{p}}{p_0}$$

Reversible trading in a closed economy

$$\begin{aligned}\text{Ext. speculator's profit} &= -\int p_0 \left(dx_0^1 + dx_0^2\right) \\ &= \int \left(\bar{p}^1 - \bar{p}^2\right) \cdot d\bar{x}^1 \\ &= p_0 \Delta \left(S_{QL}^1 + S_{QL}^2\right)\end{aligned}$$



But S_{QL} is a state variable!
Same for rev. and irrev. trade

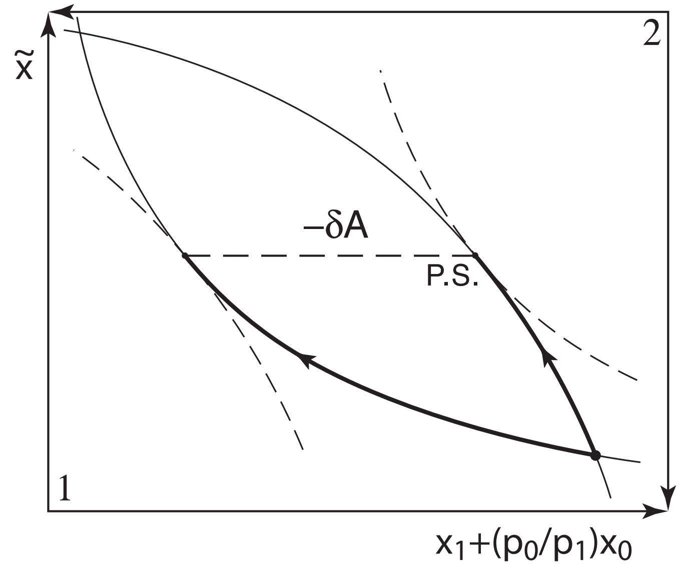
Money-metric value of trade is the amount agents could keep an external speculator from extracting

Profit extraction potentials in partially open systems

$$x \equiv (x_0, x_1, \tilde{x})$$

$$\frac{e - p_0 \mathcal{U}}{p_1} = x_1 + \frac{\tilde{p} \cdot \tilde{x}}{p_1} - \frac{p_0}{p_1} \bar{u}(x_1, \tilde{x})$$

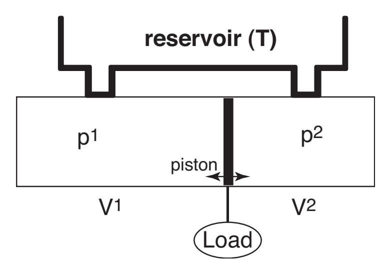
Economic "Helmholtz" potential



$$\mathcal{A}_{\mathrm{QL}} = x_1 - \frac{p_0}{p_1} \bar{u}(x_1, \tilde{x})$$

$$d\mathcal{A}_{\mathrm{QL}} = -\frac{\tilde{p}}{p_1} \cdot d\tilde{x}$$

$$-\left.\frac{\partial A}{\partial V}\right|_T = p$$



Aggregatability and “social welfare” functions

- QL economies are the most general aggregatable economies independent of composition or endowments

(Obvious reason: dual offer prices are now meaningful constraints on trading behavior)

- For these, a “social welfare” function is the sum of economic entropies
- Such economies are mathematically identical to classical thermodynamic systems

A small worked example

The dividend-discount model of finance

Contract

$$\delta M = -p_N \delta N + \frac{1}{r\delta t}\delta D$$

Energy Conservation

$$\delta U = -p\,\delta V + \delta Q$$

Constant Absolute Risk Aversion (CARA) utility model

$$\mathcal{U} \equiv N\bar{d}\left(1 - \frac{N\bar{d}}{2\nu}\sigma^2\right) - D + \phi(M)$$

$$(x_0, x_1, x_2) \equiv (-D, M, N)$$

$$(p_0, p_1, p_2) \equiv (1/r\delta t, 1, p_N)$$ think $$(T, 1, p)$$

The state-variable description

Economic entropy and basis for the social welfare function

$$S \equiv \mathcal{U} + D = N\bar{d}\left(1 - \frac{N\bar{d}}{2\nu}\sigma^2\right) + \phi\left(M\right) \qquad r\delta t = \frac{d\phi}{dM} = \left.\frac{\partial S}{\partial M}\right|_N$$

Economic "Gibbs" part of the expenditure function

$$\mathcal{G} = M + p_N N - \frac{1}{r\delta t}S \qquad \left.\frac{\partial \mathcal{G}}{\partial p_N}\right|_{r\delta t} = N$$

Economic "Helmholtz" potential for trade at fixed interest

$$\mathcal{A} = M - \frac{1}{r\delta t}S \qquad \left.\frac{\partial \mathcal{A}}{\partial N}\right|_{r\delta t} = -p_N$$

Summary comments

- Irreversible transformations are not generally predictable in either physics or economics by theories of equilibrium
- They require a theory of dynamics
- The domain in which equilibrium theory has consequences is the domain of reversible transformations
- In this domain the natural interpretation of neoclassical prices may be different

Further reading

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