Fearful Symmetries: An Introduction to Quantum Algorithms

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Physics

Fig. 1: Nature

Problems:

- come from Nature
- have solutions that are as simple, symmetric, and beautiful as possible (far more so than we have any right to expect)

Computer Science

Problems:

- are artificial
- are maliciously designed to be the worst possible
- may or may not have elegant solutions...
- ...or proofs (cf. Erdős)

Fig. 2: The Adversary

Beauty is Truth, Truth Beauty

In 1928, Dirac saw that the simplest, most beautiful equation for the electron has *two* solutions.

Four years later, the positron was found in the laboratory.

Conservation is Symmetry

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p} \quad , \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x}$$

perhaps you are more familiar with $p = mv$ and $F = ma$; try with $\mathcal{H} = (1/2)mv^2 + V(x)$

Conservation of momentum follows from translation invariance: moving entire world by dx doesn't change energy

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x} = 0$$

Conservation is Symmetry

Noether's Theorem:
symmetry implies conservation

$$\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial J} \quad , \quad \frac{dJ}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta}$$

Conservation of angular momentum
follows from symmetry under rotation!
In classical and quantum mechanics,
all conservation laws are of this form.

Relativity is Symmetry

Physics is invariant under changes of coordinates to a moving frame:

$$\begin{pmatrix} x \\ ct \end{pmatrix} \rightarrow \gamma \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

at small velocities, Galileo:

$$x \rightarrow x - vt\ ,\ t \rightarrow t$$

Groups

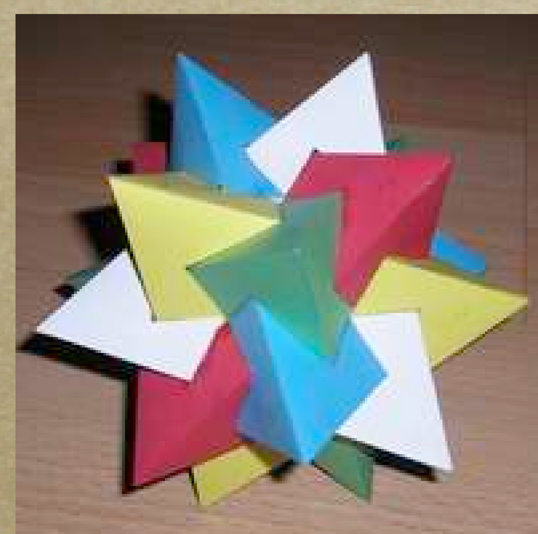
A *group* is a mathematical structure with:

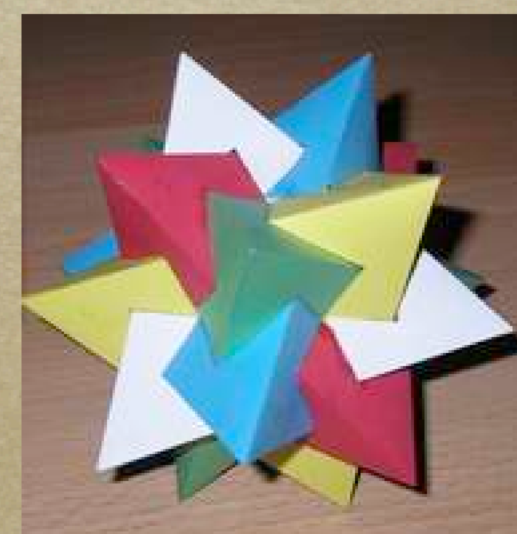
- associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- identity: $a \cdot 1 = 1 \cdot a = a$
- inverses: $a \cdot a^{-1} = a^{-1} \cdot a = 1$
- but not necessarily $a \cdot b \neq b \cdot a$

(these are *non-Abelian* groups)

Some Common Groups

- cyclic: $\mathbb{Z}_n$ (addition mod n), $\mathbb{Z}_n^*$ (multiplication)
- symmetric group (permutations): S_n
- invertible matrices
- rotations: $O(3)$
- $O(3)$ contains S_5 !

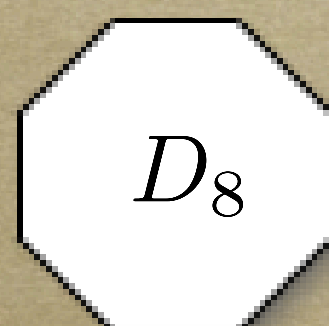




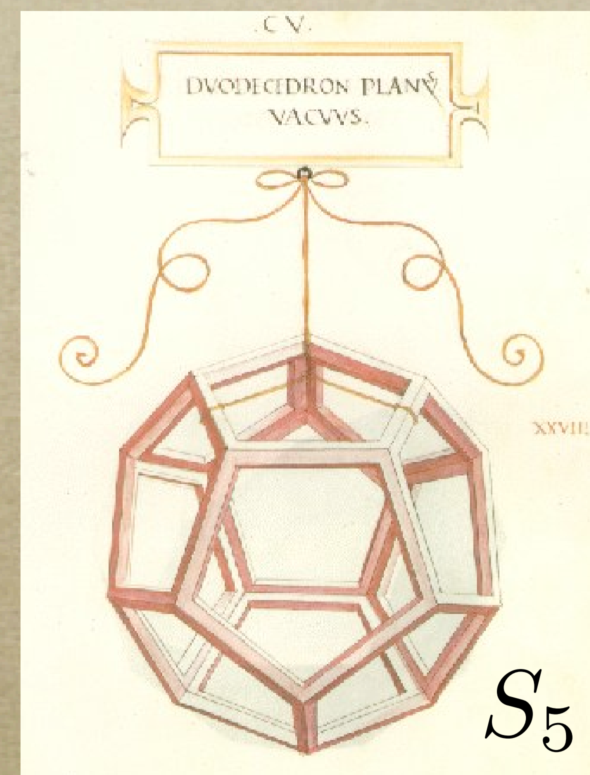
Symmetry Groups

Transformations that leave an object fixed:

$\mathbb{Z} \times \mathbb{Z}$



D_8



S_5

When Symmetry is Periodicity

- Given a function $f : \mathbb{Z}_n \to S$ we can ask for which h we have

$$f(x) = f(x+h)$$

for all x.

- These h are multiples of the periodicity r.
- The set of all such h forms a *subgroup*.

Periodicity Gives Factoring!

- To factor n, let $f(x) = c^x \mod n$.
- Find smallest r such that $f(x) = f(x+r)$ *i.e.*, $c^r \equiv 1 \mod n$. Suppose r is even:

$$c^r - 1 = kn = (c^{r/2} + 1)(c^{r/2} - 1)$$

- Now take g.c.d. of n with both factors (easy).
- Works at least 1/2 the time with random c!

Factoring: An Example

- Let's factor 15. Choose *c*=2:

$$\begin{array}{lccccccccc} x: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2^x: & 1 & 2 & 4 & 8 & 1 & 2 & 4 & 8 & 1 \end{array}$$

$$2^4 - 1 = 15 = (2^2 - 1)(2^2 + 1) = 3 \times 5$$

- Bad news: in general *r* could be as large as *n*, *i.e.*, exponentially big as a function of #digits.

Quantum Measurements

Measure *f(x)*, and "collapse" to a superposition

$$\begin{array}{llllllllll} x: & 0 & 1 & \mathbf{2} & 3 & 4 & 5 & \mathbf{6} & 7 & 8 \\ 2^x: & & & 4 & & & & 4 & & \end{array}$$

This is a random *coset* of the subgroup *H*.

But, if we simply measure *x*, all we see is a random value! This is the wrong measurement.

The Fourier Transform

Periodicities are peaks in $\hat{f}$, where ($\omega = \mathrm{e}^{2\pi i/n}$)

$$f(x) = \frac{1}{\sqrt{n}} \sum_k \hat{f}(k)\,\omega^{kx}\,,\ \hat{f}(k) = \frac{1}{\sqrt{n}} \sum_x f(x)\,\omega^{-kx}$$

Change of basis $Q_{x,k} = \dfrac{1}{\sqrt{n}}\,\omega^{kx}$

from *x* to *k*. This transformation is *unitary:*

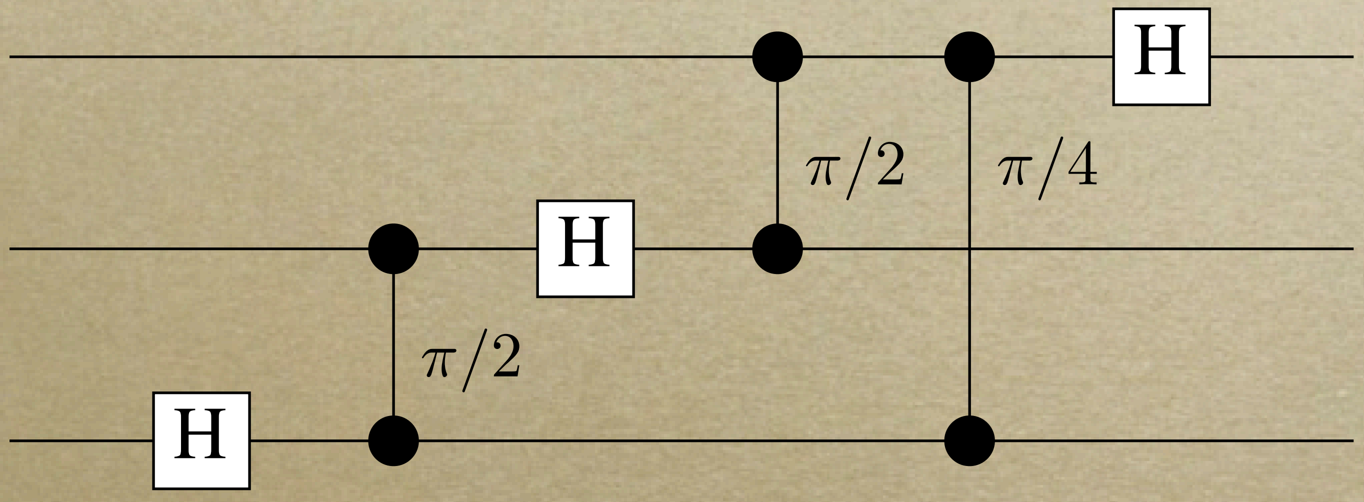
$$Q^{-1} = Q^\dagger$$

Shor's Algorithm

- Quantum mechanics allows us to perform unitary transformations.
- We can "do" the Fourier transform mod *n* with only $O(\log^2 n)$ elementary quantum operations.
- We then measure the frequency, this gives us the periodicity of *f(x)*.

Efficient Circuits for the QFT

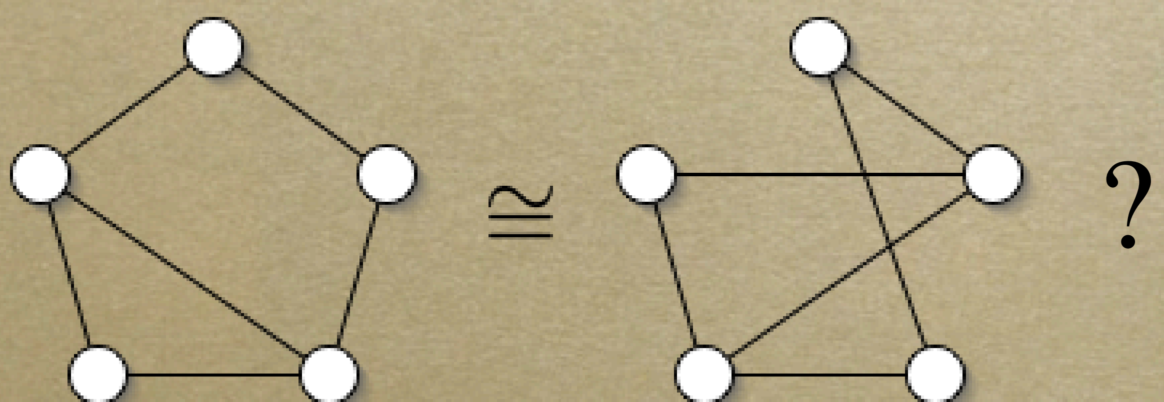
- We can break down the QFT recursively (like the FFT) into elementary gates:



- Quadratic in the number of qubits
- Thus *n* can be exponentially large!

Graph Isomorphism

- Factoring appears to be outside P, but not NP-complete. (Indeed, we believe that BQP does not contain all of NP.)
- Another candidate problem in this range:



Solving with Symmetry

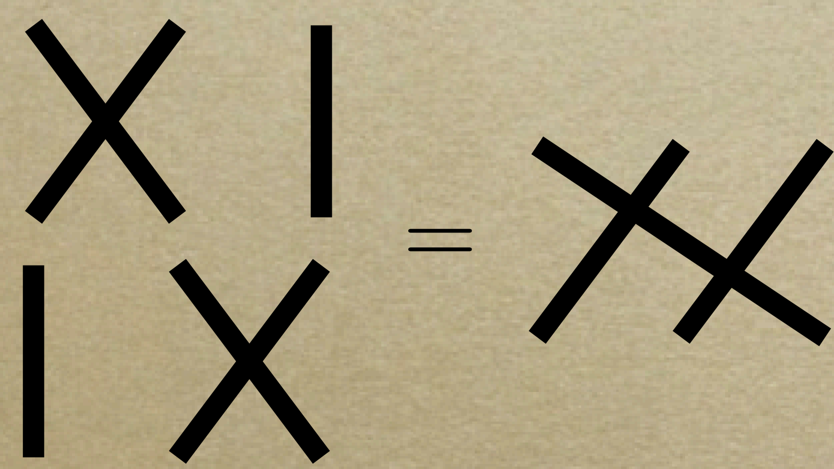
- Take the union of the two graphs. Permuting the $2n$ vertices defines a function f on S_{2n} . What is its symmetry subgroup H?
- Assume no internal symmetries. Then either f is 1-1 and $H = \{1\}$, or f is 2-1 and

$$H = \{1, m\}$$

for some m that exchanges the two graphs.

The Permutation Group

- The set of *n!* permutations of *n* things forms the permutation group S_n :



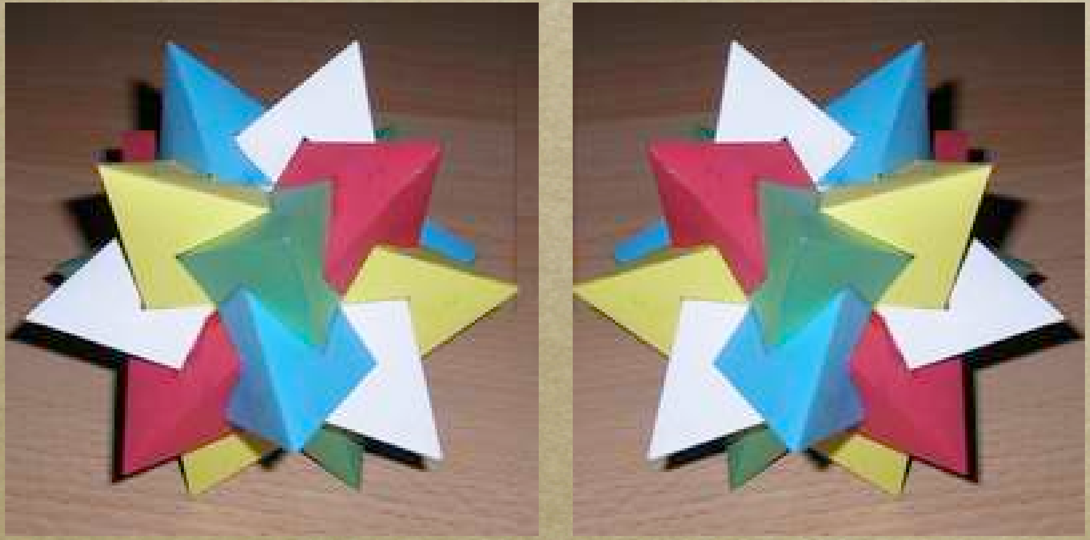
- A richly non-Abelian group ($ab \neq ba$.)

The Hidden Subgroup Problem

- We have a function $f : G \to X$
- We want to know its symmetries $H \subseteq G$
- Essentially all quantum algorithms that are exponentially faster than classical are of this form:
  - $\mathbb{Z}_n^*$ = factoring
  - S_n = Graph Isomorphism
  - D_n = some cryptographic lattice problems

Non-Abelian Fourier Transforms

- For non-Abelian G, we need *representations:*
- Geometric pictures of G in d-dimensional space



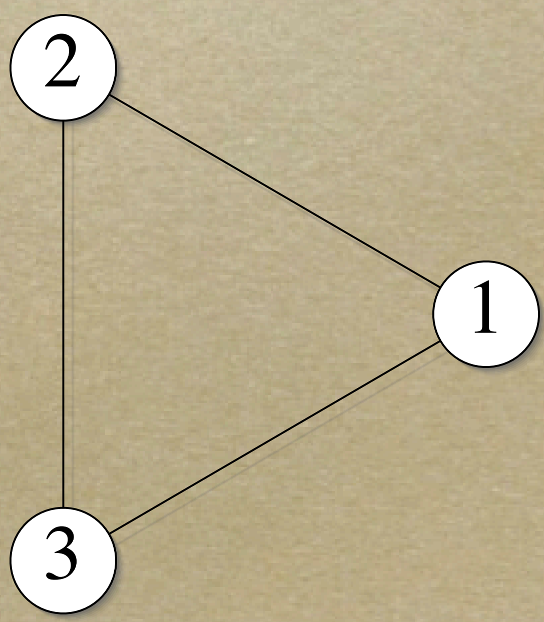
- S_5 has a three-dimensional representation: permute the colors by rotating.

Non-Abelian Fourier Transforms

- S_3 has $\mathbf{1}$ (trivial), $\pi = \pm 1$ (parity), and rotations of three points in the plane:

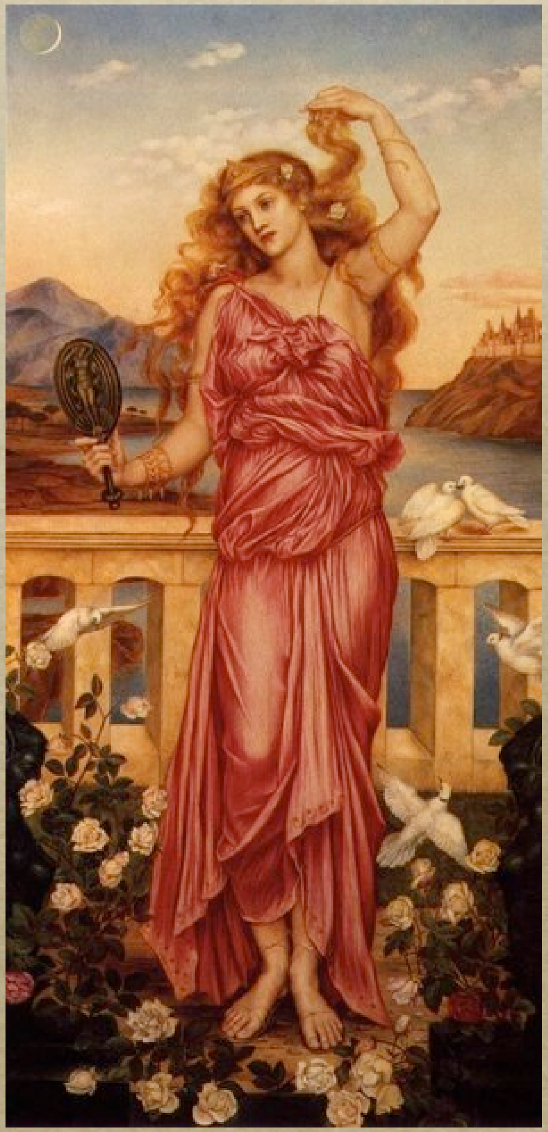
$$\rho((1\,2)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \rho((1\,2\,3)) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

- Gives 1+1+4 = 6 “frequencies,” just enough. Coincidence?



Heartbreaking Beauty

- For any group, there is a finite number of *irreducible* (“prime”) representations
- These allow us to define a Fourier transform over that group.
- Everything beautiful is true...



The Story So Far...

- It turns out that this naïve generalization of Shor's algorithm doesn't work: the permutation group S_n is "too non-Abelian."
- Tantalizingly, we know a *measurement* exists, but we don't know if we can do it efficiently.
- How much can quantum computing really do? How "special" is factoring?