Homework 6 Solutions

May 5, 2004

1 Question 12.1

A is a 202x202 matrix. with $||A||_2 = 100$ and $||A||_F = 101$. Give the sharpest possible bound on the 2-norm condition number $\kappa(A)$.

We know that

$||A||_2 = \sigma_1$ and $||A||_F = \sqrt{\sigma_1^2 + ... + \sigma_n^2}$

Hence,

$||A||_F = \sqrt{10000 + ... + \sigma_{202}^2}$

$\implies 101 = \sqrt{10000 + ... + \sigma_{202}^2}$

$\implies 201 = \sigma_2^2 + ... + \sigma_{202}^2$

$\implies$ the greatest value σ_{202} can take is 1.

$\implies \sigma_1/\sigma_{202} \geq 100$

2 Question 12.2a

Let the Vandermonde matrix of x be V_x.

Let the Vandermonde matrix of y be V_y truncated to be m x n.

Say that,

$$V_x C = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{bmatrix}$$

i.e.

$$V_x C = D \tag{1}$$

similarly,

$$V_y C = F \tag{2}$$

we wish to find A such that $AD = F$ where F is $P(y_i)$.

From eqn (1),

$$C = V_x^{-1} D \tag{3}$$

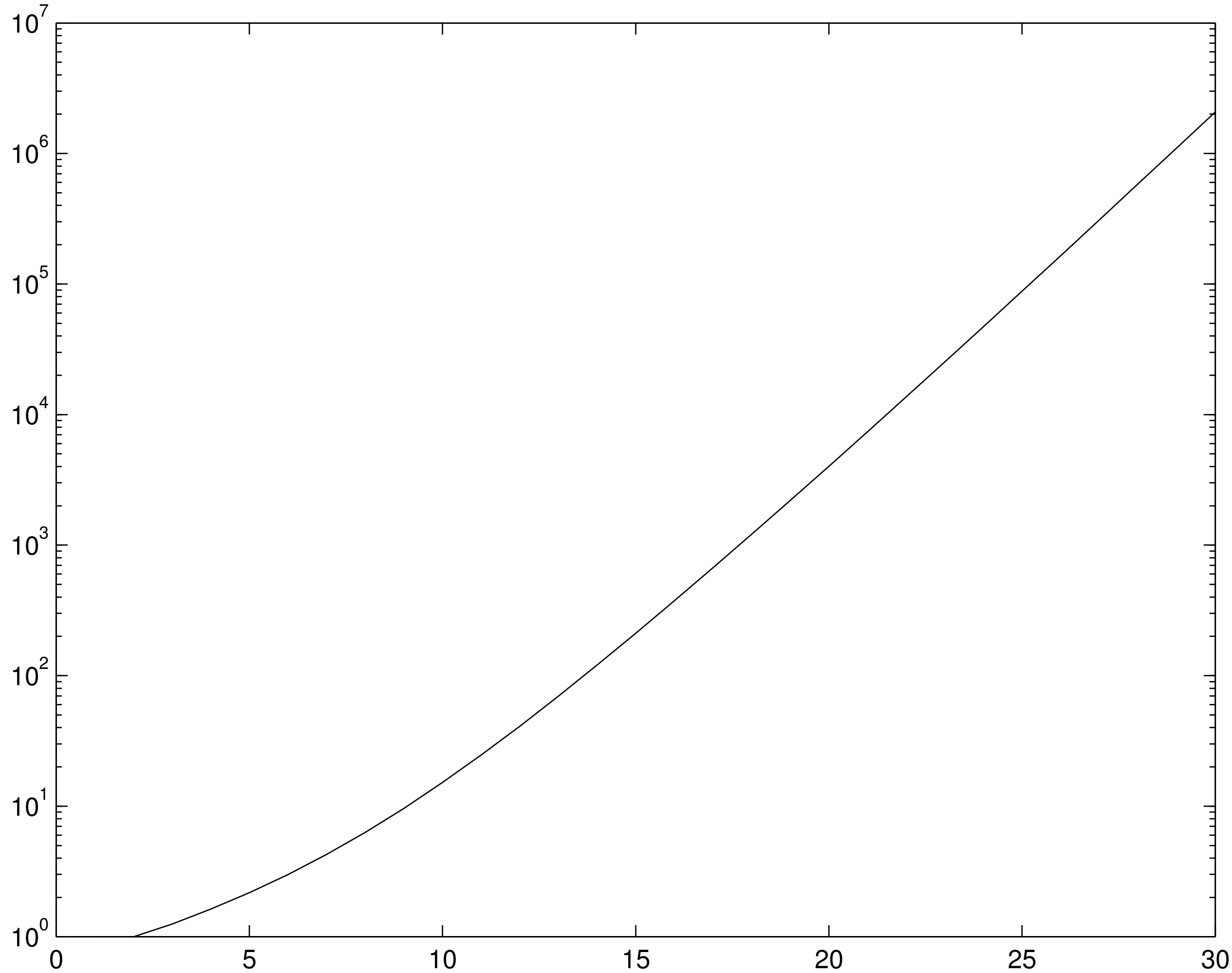
from eqns (2) and (3) we know that

$$F = V_y V_x^{-1} D$$

Hence

$$A = V_y V_x^{-1}$$

3 Question 12.2b




```
clf
norminf=[];
for i=1:30
    X=linspace(-1,1,i);
    Y=linspace(-1,1,2*i-1);

    Vx=fliplr(vander(X))
    Vy=fliplr(vander(Y));
    Vy=Vy(:,1:i);

    A=Vy*inv(Vx);
    norminf = [norminf norm(A,inf)];
end
semilogy(norminf);
```

4 Question 12.2c

Continuing from 12.2a, we know that

$V_x C = D$

Given that the function to be interpolated is the constant function 1.

Hence,

$$V_x C = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

Setting c_0 to 1 and rest to 0,

$$V_y C = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

hence $||Ax||_\infty = 1$ and $||x||_\infty = 1$

hence from 12.6

$$\frac{||J(x)||_\infty}{||Ax||_\infty / ||x||_\infty} = ||J(x)||_\infty = ||A||_\infty$$

hence $\kappa = ||A||_\infty$

Values for infinity norm contition number κ for $n = 1..30$ and $m = 2n - 1$ are:

```
1.0e+06 *   0.00000100000000
1.0e+06 *   0.00000100000000
1.0e+06 *   0.00000125000000
1.0e+06 *   0.00000162500000
1.0e+06 *   0.00000217187500
1.0e+06 *   0.00000299218750
1.0e+06 *   0.00000426367188
1.0e+06 *   0.00000629394531
1.0e+06 *   0.00000961932373
1.0e+06 *   0.00001518344116
1.0e+06 *   0.00002466098785
```

```
1.0e+06 *   0.00004104731369
1.0e+06 *   0.00006973739958
1.0e+06 *   0.00012050918078
1.0e+06 *   0.00021118414587
1.0e+06 *   0.00037440904610
1.0e+06 *   0.00067026320675
1.0e+06 *   0.00120977020274
1.0e+06 *   0.00219887390852
1.0e+06 *   0.00402091400412
1.0e+06 *   0.00739169457486
1.0e+06 *   0.01365172154724
1.0e+06 *   0.02531814084958
1.0e+06 *   0.04712928061361
1.0e+06 *   0.08802518063473
1.0e+06 *   0.16490944740095
1.0e+06 *   0.30980701131354
1.0e+06 *   0.58350692084493
1.0e+06 *   1.10159808788373
1.0e+06 *   2.08411298423162
```

5 Question 12.2d

At $n = 11$, the value of κ is 24.67. From, fig 11.1, we see that the bound is approximately $= 4$.

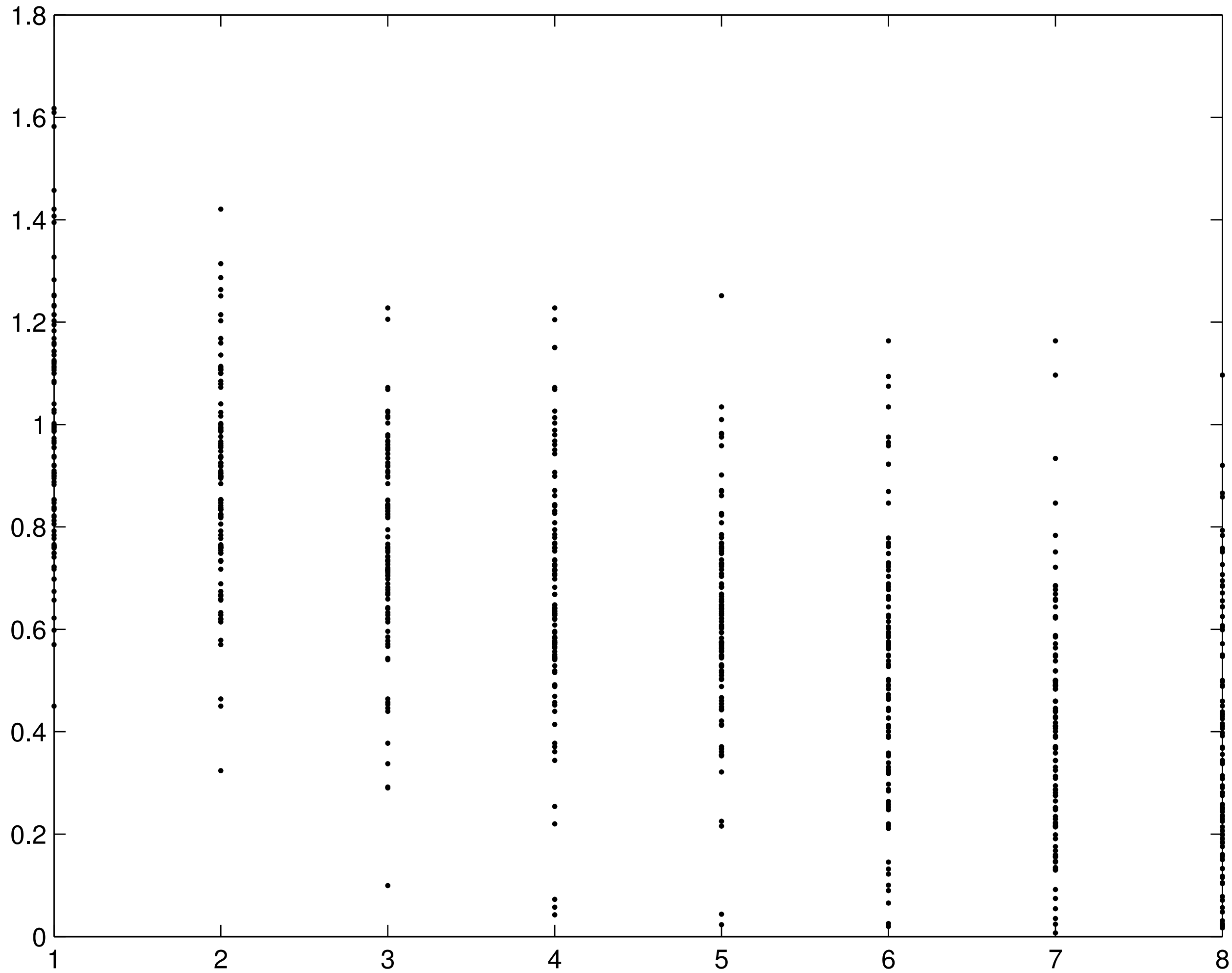
Hence we are not near the implicit bound.

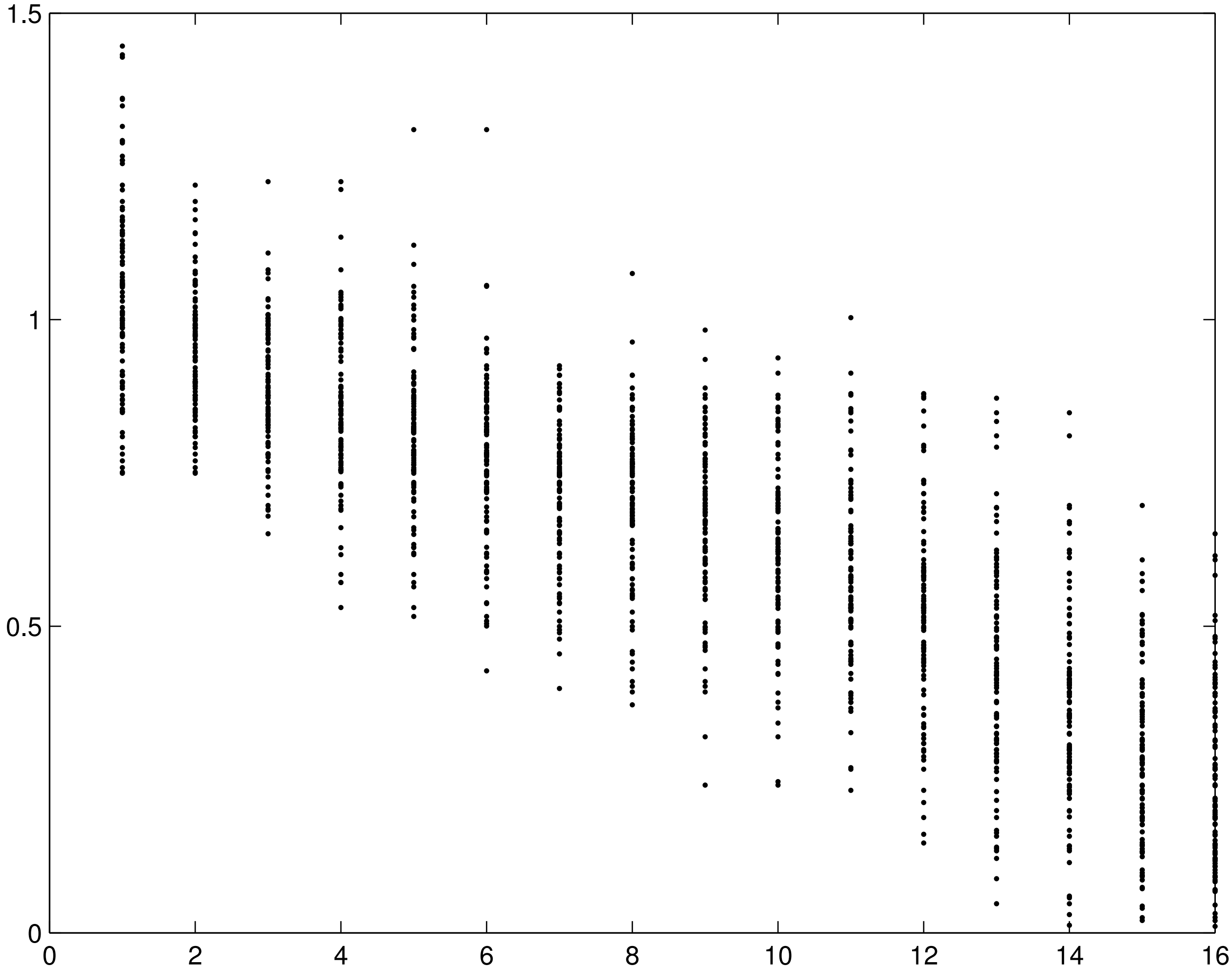
6 Question 12.3a

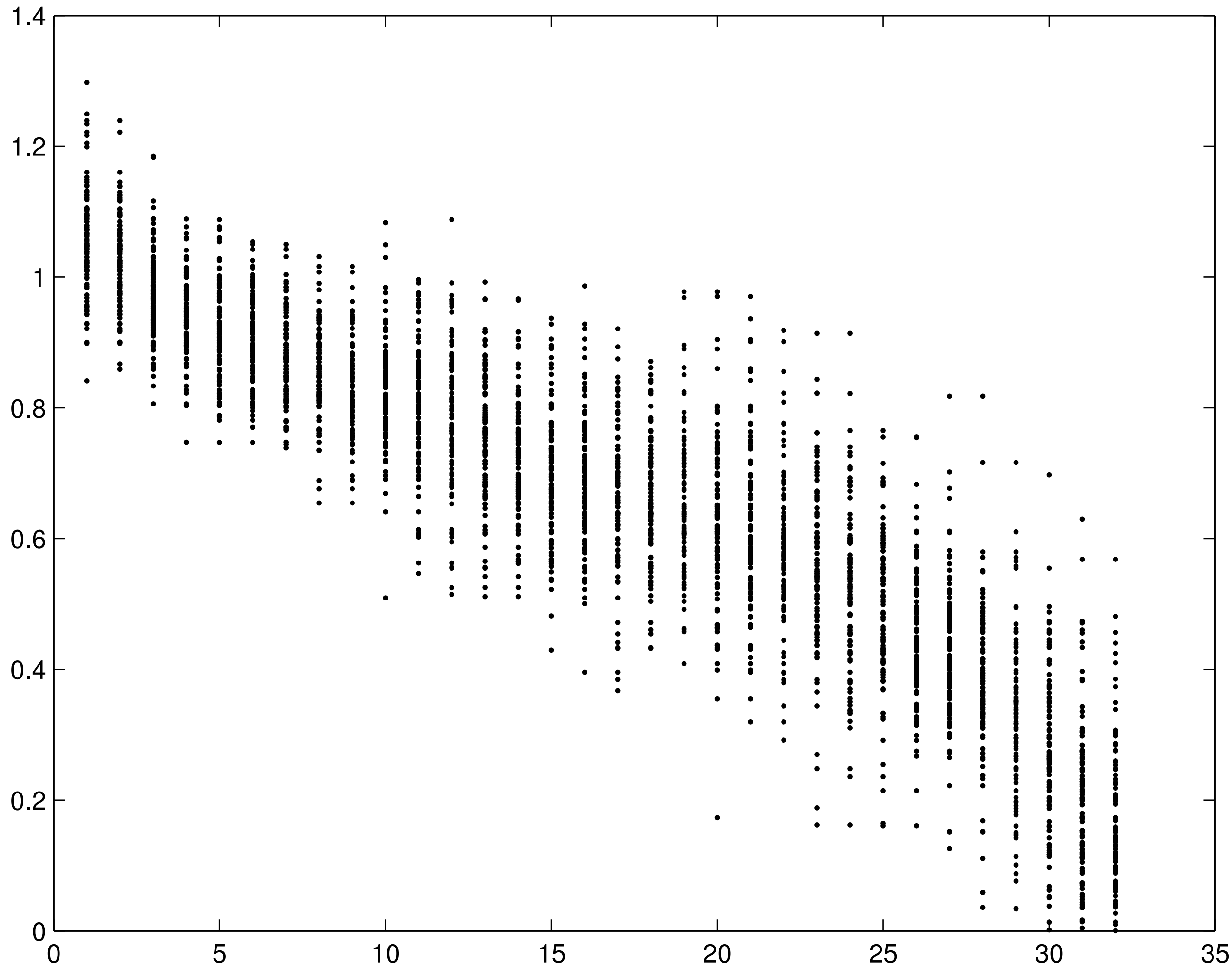
```
spec_mean=zeros(4,1);

for j=3:6
    figure;
    spec=zeros(100,1);
    m=2.^j;
    eigs = zeros(100,m);
    for i=1:100
        A = randn(m,m)/sqrt(m);
        [c,d]=eig(A);
        eigs(i,:)=diag(d)';
        plot([1:m],abs(eigs(i,:)),'.');
        spec(i) = max(abs(eigs(i,:)));
        hold on;
```

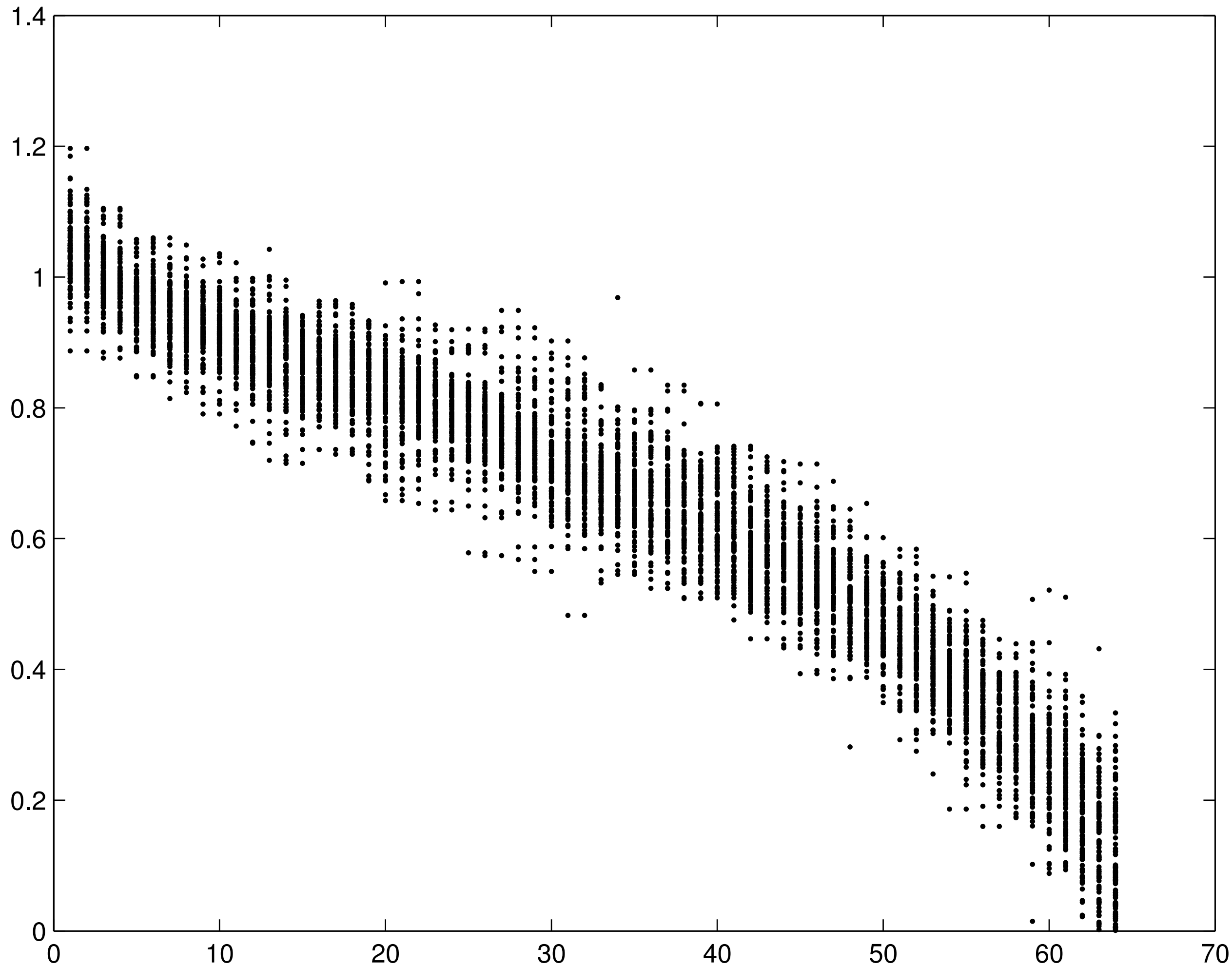
```
    end
    spec_mean(j-2) = mean(spec);
end
figure;
length(spec_mean)
plot(2.^[3:6],spec_mean,'r-');
```

The following graphs are the results of the above program.

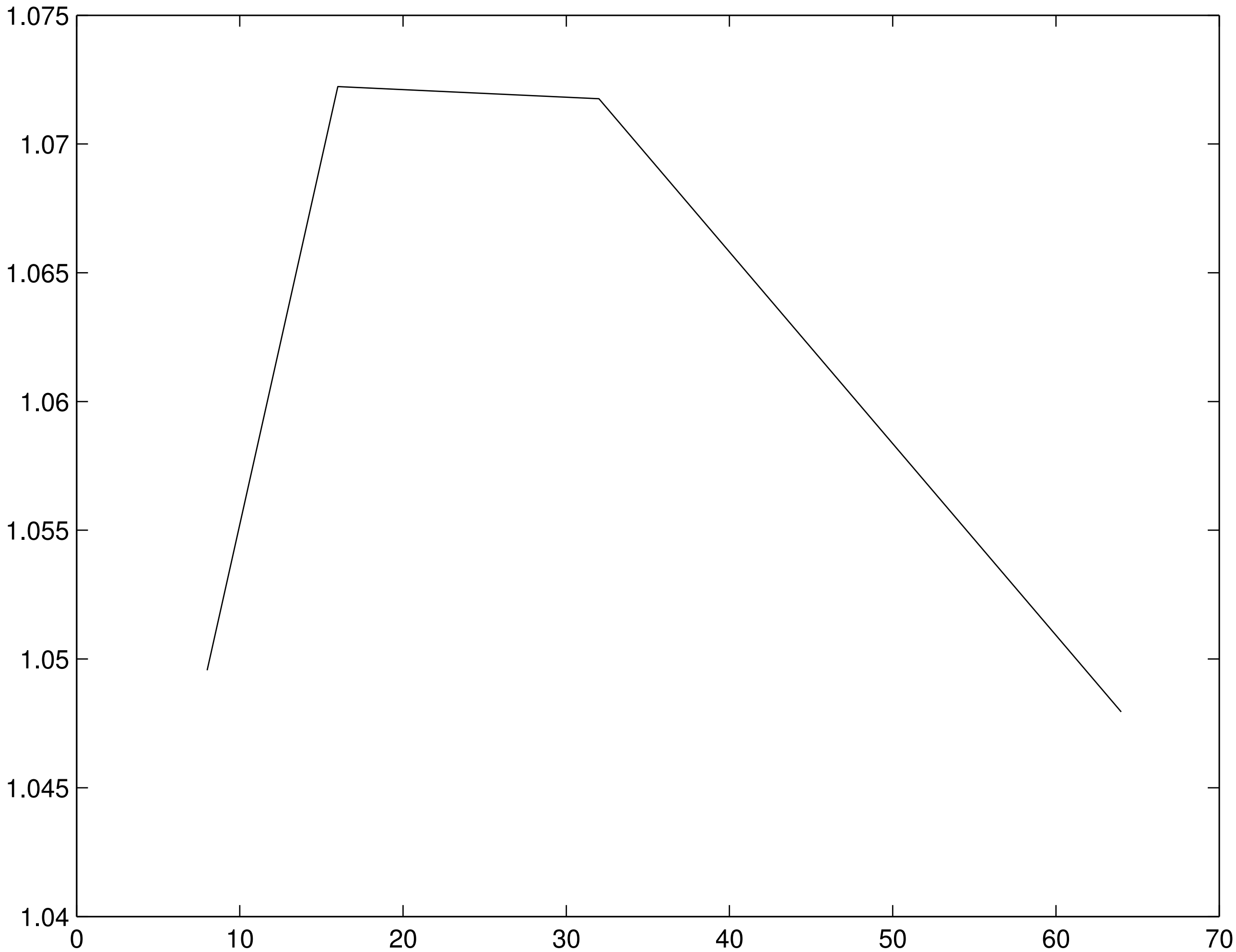



1.5
1
0.5
0
0
2
4
6
8
10
12
14
16


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35



The mean spectral radius varies as shown in the following graph:



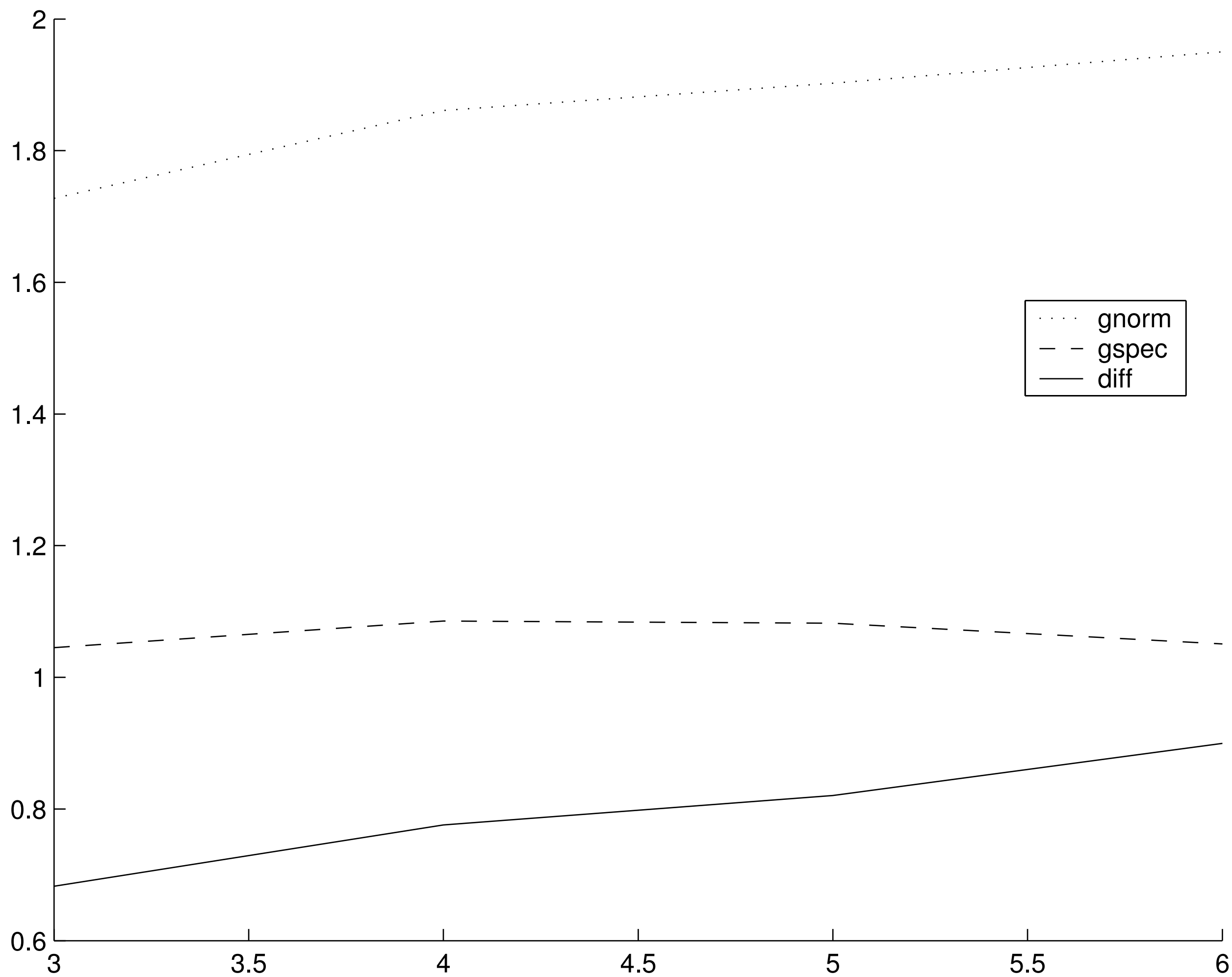
Interestingly, there is no pattern to the behaviour of the spectral radius for a normally distributed random matrix. However if the matrix is a uniformly distributed random matrix, then the graph shows an increasing pattern in the spectral radius.

7 Question 12.3b

```
j=1;
clf;
allNorms=[];
allSpecs=[];
idx =3:6;
for j=idx
gnorm=[];
gspec=[];
for i=1:100
```

```
m=2^j;
A = randn(m,m)/sqrt(m);
gnorm = [ gnorm norm(A)];
gspec = [ gspec max(abs(eig(A)))];

hold on;
end
allNorms=[allNorms mean(gnorm)];
allSpecs=[allSpecs mean(gspec)];
end
plot(idx,allNorms,'r:');
plot(idx,allSpecs,'b--');
plot(idx,allNorms-allSpecs,'k-');
legend('gnorm','gspec','diff',0);
```

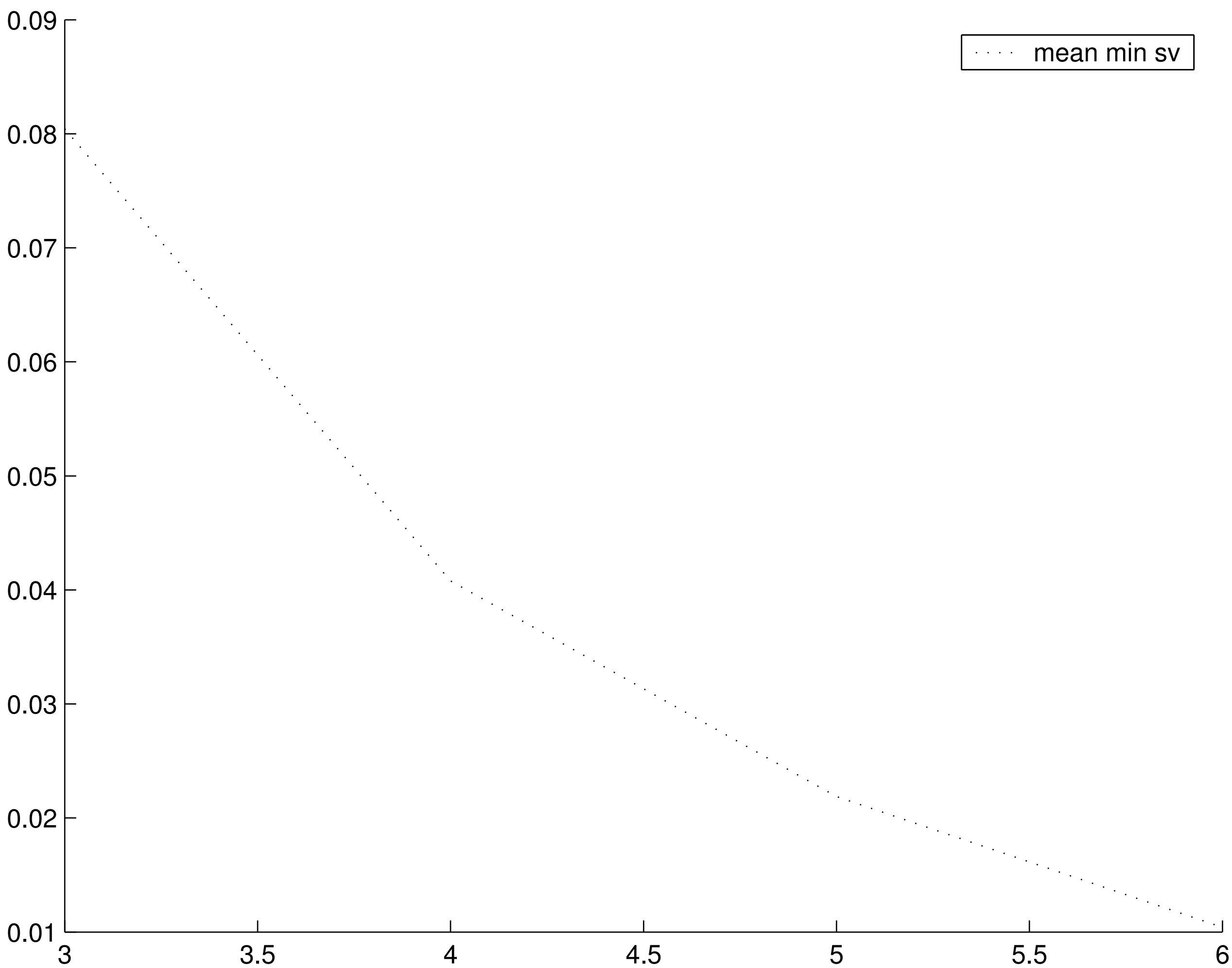
The following figure is an example of the output of the above program:



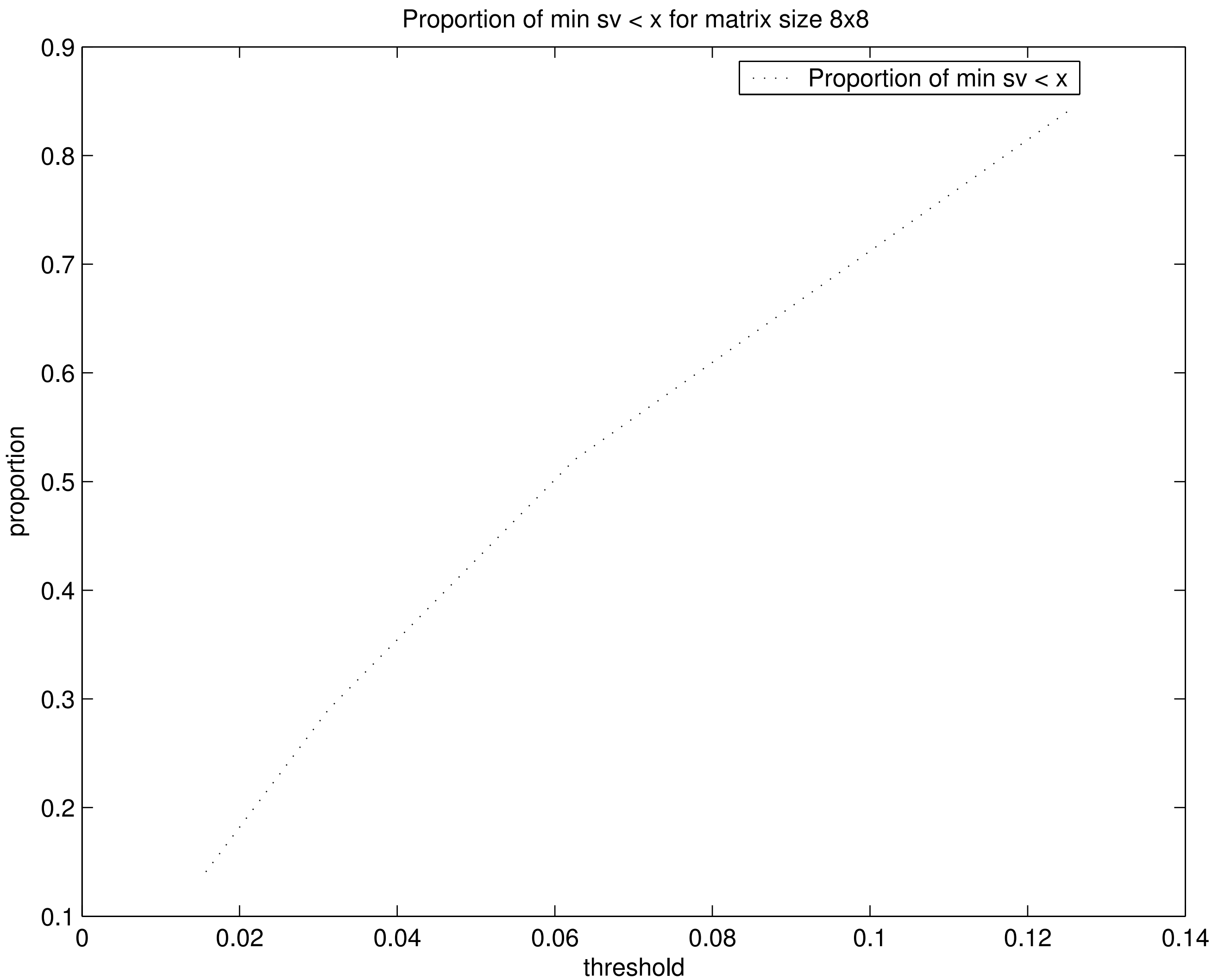
For a normally distributed random matrix, ρ does not approach $||A||$ as m approaches infinity. However it does for uniformly distributed random matrices.

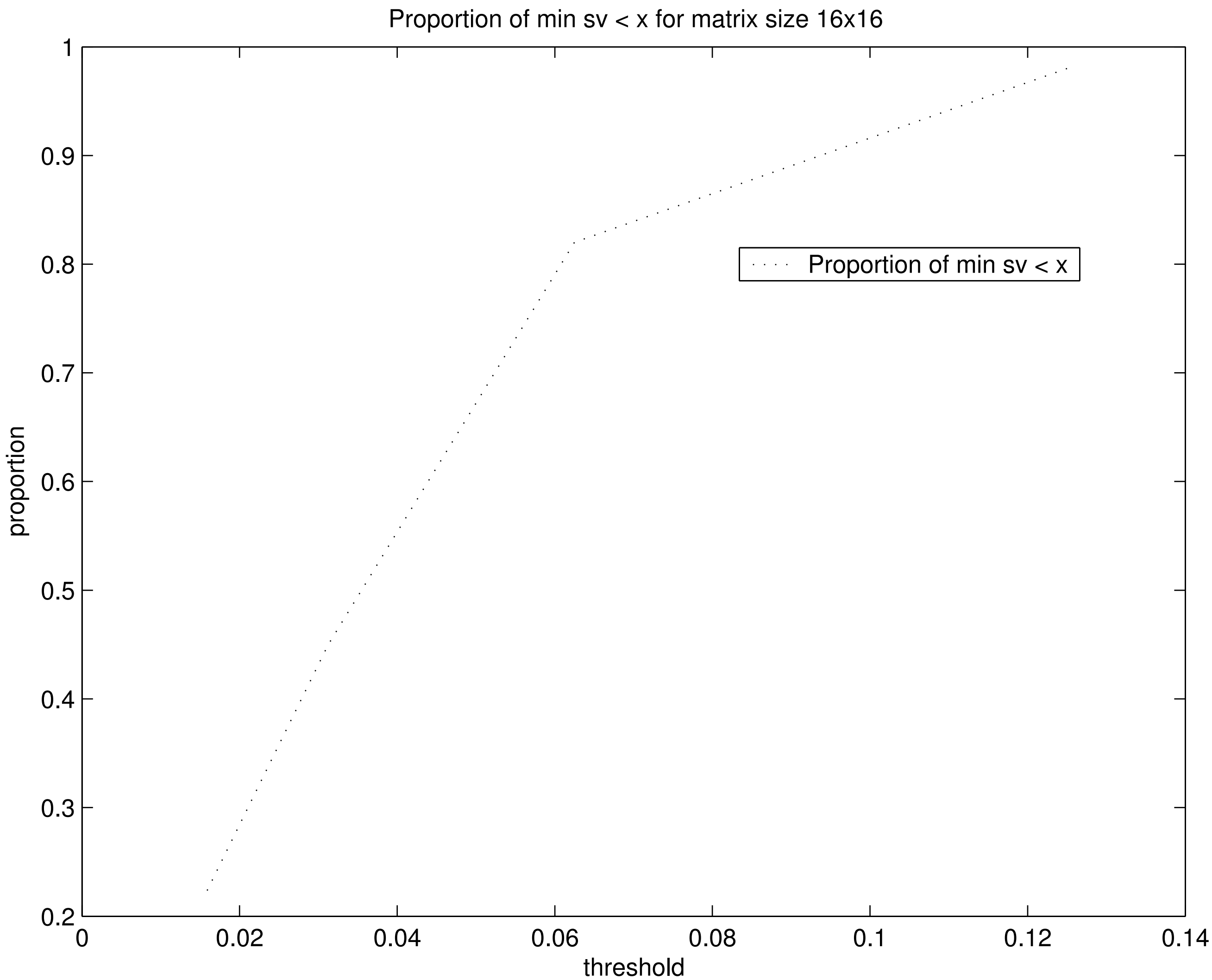
8 Question 12.3c

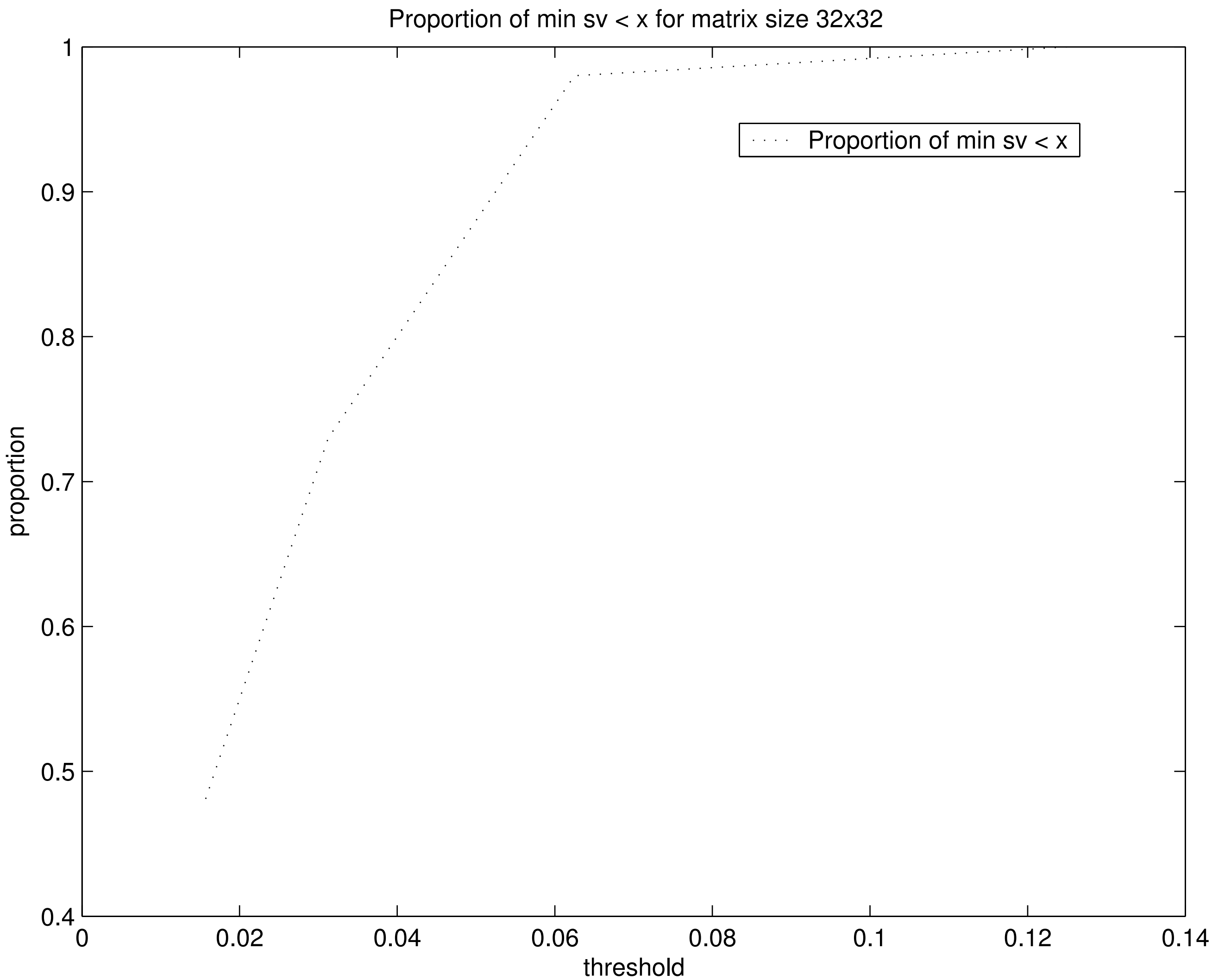
The σ_{min} value decreases with increasing matrix size.



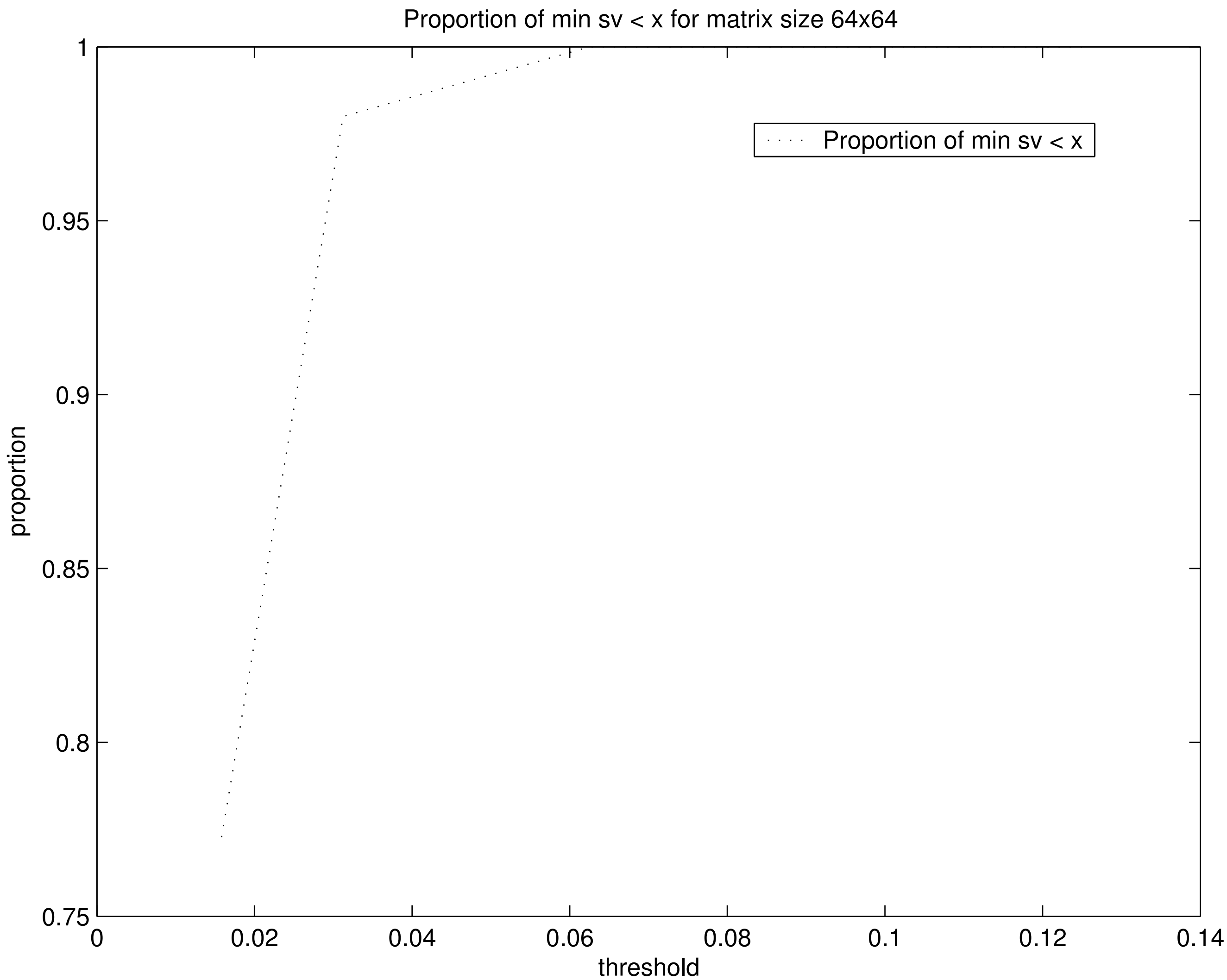
The following graphs denote the proportion of σ_{min} below 8^{-1}, 16^{-1}, 32^{-1} and 64^{-1}.


Proportion of min sv < x for matrix size 8x8
Proportion of min sv < x
proportion
threshold
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14


Proportion of min sv < x for matrix size 16x16
proportion
threshold
Proportion of min sv < x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14



Proportion of min sv < x for matrix size 32x32
Proportion of min sv < x
proportion
threshold
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14



9 Question 12.3c

The answers to (a)–(c) change in only one way. The behaviour of the curves plotted is even *more* marked but in the same direction as the previous plots.

10 Question 18.1a

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 10001 & -5000 & -5000 \\ -10000 & 5000 & 5000 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

11 Question 18.1b

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 \\ 2.0001 \\ 2.0001 \end{bmatrix}$$

12 Question 18.1c

$\kappa(A) = 4.242923541617029e+04$

$\theta = 0.68470287326118$

$n = 1.00000000055554$

13 Question 18.1d

The four condition numbers are:

	y	x
b	1.29097723607894	$5.477517703608065e+04$
A	$5.477517700565104e+04$	$1.469883252863362e+09$

14 Question 18.1e

14.1

We are considering how perturbations in b affect y in the equation $y = Pb$

let us use

$$\delta b = \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix}$$

page 132 tells us how.

using this choice of δb and $\delta f = f(x + \delta x) - f(x)$

$$\delta(y) = P(b + \delta b) - P(b) = P(b) - P(b) + P(\delta b) = \epsilon P \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix} = \delta b$$

$$\implies \delta y = \delta b$$

from this and 12.5, we get

$$\frac{||\delta y||/||y||}{||\delta b||/||b||} = \frac{||b||}{||y||}$$

this is the *condition number* itself. Hence our value for δb achieves the condition number.

14.2

Now let us consider how perturbations in b affect x in equation $x = A^{+}b$

Let $U\Sigma V^*$ be the svd of A.

by equation 11.21 $A^{+} = V\Sigma U^*$

Let $\delta b = \epsilon u_n$ where u_n is the left singular vector corresponding to the smallest singular value of A.

form of 12.5 is now,

$$\frac{||\delta x||/||x||}{||\delta b||/||b||}$$

$$\delta x = A^+(b + \delta b) - A^+(b) = A^+(\delta b) = \epsilon A^+(u_n) = \epsilon V \Sigma^{-1} U^* u_n = \epsilon v_n \sigma_n^{-1} = \epsilon \frac{v_n}{\sigma_n}$$

$$||\delta b|| = ||\epsilon u_n|| = \epsilon ||u_n|| = \epsilon$$

$$||\delta x|| = \epsilon ||\frac{v_n}{\sigma_n}|| = \frac{\epsilon}{\sigma_n}$$

We get,

$$\frac{||\delta x||/||x||}{||\delta b||/||b||} = \frac{||b||}{||x||\sigma_n}$$

substituting we get,

$$= 5.477517703596802e + 04$$

hence $\delta b = \Sigma u_2$ achieves the condition number.