CDC99-REG0435

Design of Optimal Gains for the Proportional Integral Kalman Filter with application to Single Particle Tracking

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March 1, 1999

1 Introduction

Kalman filtering is an optimal state estimation process applied to a dynamic system that involves random perturbations. It has found wide application in areas such as stochastic optimal control and radar tracking. However, application of standard Kalman Filter requires an accurate model of the process under consideration and if there are errors in the process model, the Kalman filter may lead to poor performance.

The concept of integral action was used in observer theory [1, 2] for the purpose of improving recovery in LTR design. The PI Kalman filter, which modifies the Kalman filter with an integral term, was introduced first in [3] and further developed in [4]. In this recent work, the ability of the integral action to estimate and compensate effectively both plant perturbations and input disturbances was demonstrated. It has been further shown that, a modified version of the filter with a fading integral is also capable of coping transitory effect and improve the stability margin. In these works, the proportional gain was designed using the Kalman filter procedure and the integral gain was designed to ensure system stability, through a trial and error process.

While the PI Kalman filter is established as a powerful tool, a systematic approach for the design of the gains are necessary. The design of the PI Kalman filter in its most general form involves the design of four matrices: The proportional and integral gains, the fading constant, and the integral effect coefficient. Design of the proportional and integral Kalman gains is of particular importance in order to preserve the optimality property of the Kalman filter.

In this work we introduce an optimal design strategy for the PI Kalman filter. Both the proportional and integral gains are selected optimally. Moreover, guidelines for the design of the fading constant and the integral effect coefficient are discussed. Once the fading constant and the integral effect coefficient are selected using performance specifications, the suggested algorithm can be used to obtain minimum variance estimates.

The suggested algorithm was tested as a single particle tracking system, with a maneuvering target. The results were compared with those of a conventional Kalman filter (P Kalman filter). It has been observed that the PI Kalman filter is especially beneficial if the target exhibits unmodeled behavior, or the system noise is colored or is not Gaussian, contrary to the initial assumption. These are some conditions which are characteristic to many real world problems [5].

In the following section, we introduce the Proportional Integral Kalman Filter in its most general form. Section 3 covers the optimal design procedure for the PI Kalman filter, as well as a discussion of the remaining design parameters. The single particle tracking system is given in Section 4, with simulation results of the PI Kalman filter. We state our conclusions in Section 5.

2 The General PI Kalman Filter

The problem at hand is the estimation of the states of a noisy linear system from the given output vector. The system in consideration is:

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \tag{1}$$

where x is the system state vector, y is the output vector, w is white system noise with $Q(k) = E\{w^\top(k)w(k)\}$ and v is white measurement noise with $R(k) = E\{v^\top(k)v(k)\}$. The Kalman filter for this system is given as [6]

$$\hat{x}(k+1) = A\hat{x}(k) + G(y(k) - C\hat{x}(k)) \tag{2}$$

where G is updated as follows:

$$\begin{aligned} P(k-1^+) &= P(k-1) - G(k-1)CP(k-1) \\ P(k) &= AP(k-1^+)A^\top + BQ(k)B^\top \\ G(k) &= (AP(k-1^+)A^\top C^\top + BQ(k)B^\top C^\top)\times \\ &\quad (CAP(k-1^+)A^\top C^\top + R(k) + CBQ(k)B^\top C^\top)^{-1} \end{aligned} \tag{3}$$

The suggested PI Kalman filter in previous works is

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + \hat{d}(k) + G_P(y(k) - C\hat{x}(k)) \\ \hat{d}(k+1) &= \hat{d}(k) + G_I(k)(y(k) - C\hat{x}(k)) \end{aligned} \tag{4}$$

where $\hat{d}(k)$ represents the integral term. Note that $\hat{d}(k)$ is not an "estimate", the purpose of the notation will become clear in the following section. In this work, the fading constant and the integral effect coefficient are both incorporated in the above system. Therefore, the resulting filter, the generalized Proportional-Integral Kalman Filter is given in an augmented system as

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{d}(k+1) \end{bmatrix} = \begin{bmatrix} A & D_i \\ 0 & D_f \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{d}(k) \end{bmatrix} + \begin{bmatrix} G_P(k) \\ G_I(k) \end{bmatrix} (y(k) - C\hat{x}(k)) \tag{5}$$

where D_i, D_f are constant matrices to be designed prior to tracking according to performance specifications and G_P, G_I are Kalman gains. In the next section, we describe a methodology for the design of proportional, as well as integral, Kalman gains.

3 Design of PI Kalman Filter

The design of the PI Kalman filter consists of two steps. The first step is to determine the integral effect coefficient D_i and fading constant D_f using information system specifications and performance requirements. The second step is the optimal design of the proportional and integral Kalman gains, namely G_P and G_I. In this section, we introduce an optimal design procedure for Kalman gains, as well as guidelines for the design of the fading and integral effect coefficients.

The optimal design of proportional and integral gains is achieved by examining the augmented system

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k) + \bar{B}w(k) \\ y(k) &= \bar{C}\bar{x} + v(k) \end{aligned} \tag{6}$$

where

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & D_i \\ 0 & D_f \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{7}$$

Using this system, the optimal gains will be given by the following algorithm:

$$\begin{aligned} \bar{P}(k-1^+) &= \bar{P}(k-1) - G(k-1)\bar{C}\bar{P}(k-1) \\ &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{aligned} \tag{8}$$

$$\bar{P}(k) = \bar{A}\bar{P}(k-1^+)\bar{A}^\top + \bar{B}Q(k)\bar{B}^\top$$

$$G_P(k) = \left(AP_{11}A^\top C^\top + D_iP_{21}A^\top C^\top + AP_{12}D_i^\top C^\top + D_iP_{22}^\top D_i^\top C^\top + BQ(k)B^\top C^\top\right) \times \left(CAP_{11}A^\top C^\top + CD_iP_{21}A^\top C^\top + CAP_{12}D_i^\top C^\top + CD_iP_{22}^\top D_i^\top C^\top + R(k) + CBQ(k)B^\top C^\top\right)^{-1}$$

$$G_I(k) = \left(D_fP_{21}A^\top C^\top + D_fP_{22}D_i^\top C^\top\right) \times \left(CAP_{11}A^\top C^\top + CD_iP_{21}A^\top C^\top + CAP_{12}D_i^\top C^\top + CD_iP_{22}^\top D_i^\top C^\top + R(k) + CBQ(k)B^\top C^\top\right)^{-1}$$

$$G(k) = [G_P(k)^\top \quad G_I(k)^\top]^\top \tag{9}$$

The filter is initialized as follows:

$$\bar{P}(0) = \begin{bmatrix} cov\{x(0)\} & 0 \\ 0 & I \end{bmatrix}, \quad \hat{x}(0) = E\{x(0)\}, \quad \hat{d}(0) = 0 \tag{10}$$

and the estimated states are updated using (5). Note that when $D_i = D_f = 0$, i.e. there is no integral action, the proportional Kalman gain G_P reduces to (3).

As mentioned above, D_i and D_f are constant matrices that should be designed prior to operation using performance criteria. The following are some guidelines for the design of these constants that will be made clear in the results section.

- D_i defines the extent of the effect of the integral term on the estimate.
- D_i should be full rank to ensure observability of the overall system (6).
- A larger D_i results in smaller estimation error, however, it increases the overshoot problem inherent to the PI filter at start-up. The requirements of the specific system should be used to determine D_i.
- $D_f = \alpha I$ is the fading constant, $0 < \alpha \leq 1$.
- For systems with persistent perturbations, use of fading (i.e. $D_f \neq I$) is of no particular advantage. On the other hand, for systems with a low occurrence of disturbances, fading is necessary to improve stability margin.

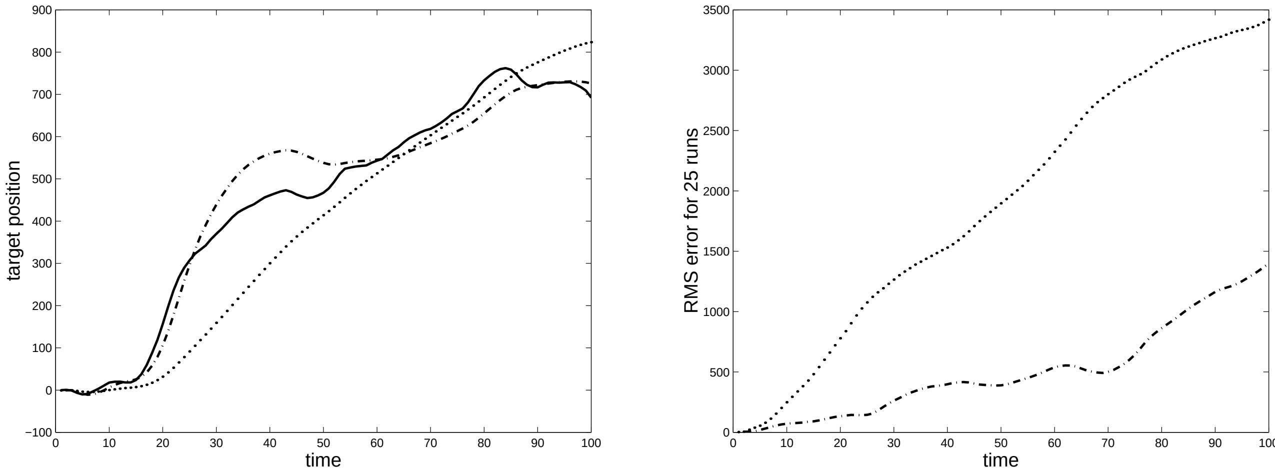

Figure 1: Left: Actual position (solid line) position estimated with PI Kalman filter (dashed line) and position estimated with P Kalman filter (dotted line) for a frequently maneuvering target. Right: Mean estimation error for PI tracker (dashed line) and P tracker (dotted line) after 25 runs.

The design of D_i and D_f are highly dependent on the nature of the system, nature of disturbances, noise levels and system requirements. Our simulations showed that while the performance of the PI Kalman filter is comparable to the P Kalman filter for systems with low noise levels, better performance can be achieved by tuning D_i and D_f properly. On the other hand, for systems with high noise levels, PI Kalman filter always performs considerably better than the P Kalman filter.

4 The Single Particle Tracking Problem

In the single particle tracking problem, we consider an object moving in the xy-plane with constant velocity. The x and y motions are assumed to be uncorrelated. Here we give equations for the x motion.

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} w \\ y(k) &= x_1(k) + v(k) \end{aligned} \tag{11}$$

At this point we should emphasize that the PI version of the Kalman filter is recommended for systems which could exhibit unexpected behavior, such as a maneuvering target.

We simulated the suggested method using MATLAB, to track a single target with random maneuvers. The results are depicted in Figure 1.

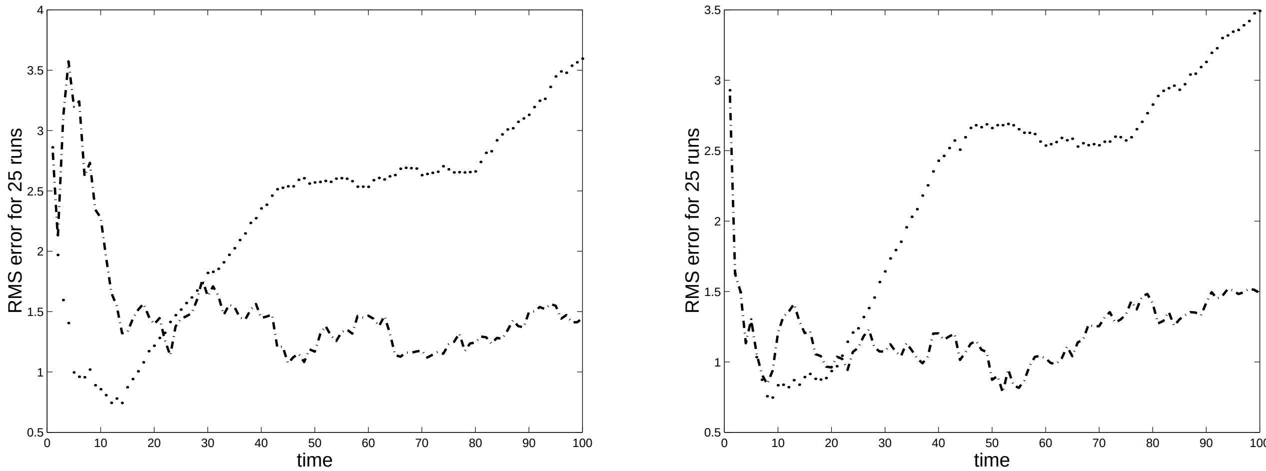

Figure 2: Effect of D_i on performance for a system with smaller noise. Both figures show the mean estimation error for the PI tracker (dashed line) and the P tracker (dotted line) for 25 runs. Left shows the performance with $D_i = I$, right shows the performance with $D_i = 0.1I$

An important goal of the simulation step was to develop and verify the design guidelines described in the previous section. To this end, we varied the design constants and obtained the following results. For this specific type of trajectory, the inclusion of fading was not of particular benefit. This is because the target is maneuvering at each instant, making the integral action important at each instant. On the other hand, the reduction of D_i reduced the error in the PI Kalman filter at startup, with the cost of increased error –though still considerably less than the P Kalman filter– at final time. These can be observed from Figure 2.

5 Conclusions and Future Work

In this work, an optimal design procedure for the proportional and integral gains in the PI Kalman filter is introduced. Guidelines for the selection of fading and integral effect constants are also given. It has been observed that the performance of the PI Kalman filter is superior to the conventional Kalman filter as a single particle tracker for a frequently maneuvering target.

An immediate extension of this algorithm will be for the multiple target tracking problem. The tracker will be based on the use of an extended version of the filter introduced in this work, modified to be used with the nonlinear symmetric measurement equations.

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