For the capacity constraint, first observe that if $(u,v) \in E$, then $c_f(v,u) = f(u,v)$. Therefore, we have $f'(v,u) \le c_f(v,u) = f(u,v)$, and hence

$$\begin{aligned}
(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) && \text{(by equation (26.4))} \\
&\ge f(u,v) + f'(u,v) - f(u,v) && \text{(because } f'(v,u) \le f(u,v)\text{)} \\
&= f'(u,v) \\
&\ge 0 .
\end{aligned}$$

In addition,

$$\begin{aligned}
&(f \uparrow f')(u,v) \\
&\quad = f(u,v) + f'(u,v) - f'(v,u) && \text{(by equation (26.4))} \\
&\quad \le f(u,v) + f'(u,v) && \text{(because flows are nonnegative)} \\
&\quad \le f(u,v) + c_f(u,v) && \text{(capacity constraint)} \\
&\quad = f(u,v) + c(u,v) - f(u,v) && \text{(definition of } c_f\text{)} \\
&\quad = c(u,v) .
\end{aligned}$$

To show that flow conservation holds and that $|f \uparrow f'| = |f| + |f'|$, we first prove the claim that for all $u \in V$, we have

$$\begin{aligned}
&\sum_{v \in V} (f \uparrow f')(u,v) - \sum_{v \in V} (f \uparrow f')(v,u) \\
&\qquad = \sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) + \sum_{v \in V} f'(u,v) - \sum_{v \in V} f'(v,u) . \qquad (26.5)
\end{aligned}$$

Because we disallow antiparallel edges in $G$ (but not in $G_f$), we know that for each vertex $u$, there can be an edge $(u,v)$ or $(v,u)$ in $G$, but never both. For a fixed vertex $u$, let's define $V_1(u) = \{v : (u,v) \in E\}$ to be the set of vertices with edges from $u$, and $V_2(u) = \{v : (v,u) \in E\}$ to be the set of vertices with edges to $u$. We have $V_1(u) \cup V_2(u) \subseteq V$ and, because we disallow antiparallel edges, $V_1(u) \cap V_2(u) = \emptyset$. By the definition of flow augmentation in equation (26.4), only vertices in $V_1(u)$ can have positive $(f \uparrow f')(u,v)$, and only vertices in $V_2(u)$ can have positive $(f \uparrow f')(v,u)$. Starting from the left-hand side of equation (26.5), we use this fact and then reorder and group terms, giving

$$\begin{aligned}
&\sum_{v \in V} (f \uparrow f')(u,v) - \sum_{v \in V} (f \uparrow f')(v,u) \\
&\quad = \sum_{v \in V_1(u)} (f \uparrow f')(u,v) - \sum_{v \in V_2(u)} (f \uparrow f')(v,u) \\
&\quad = \sum_{v \in V_1(u)} (f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v \in V_2(u)} (f(v,u) + f'(v,u) - f'(u,v))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{v \in V_1(u)} f(u,v) + \sum_{v \in V_1(u)} f'(u,v) - \sum_{v \in V_1(u)} f'(v,u) \\
&\qquad - \sum_{v \in V_2(u)} f(v,u) - \sum_{v \in V_2(u)} f'(v,u) + \sum_{v \in V_2(u)} f'(u,v) \\
&= \sum_{v \in V_1(u)} f(u,v) - \sum_{v \in V_2(u)} f(v,u) \\
&\qquad + \sum_{v \in V_1(u)} f'(u,v) + \sum_{v \in V_2(u)} f'(u,v) - \sum_{v \in V_1(u)} f'(v,u) - \sum_{v \in V_2(u)} f'(v,u) \\
&= \sum_{v \in V_1(u)} f(u,v) - \sum_{v \in V_2(u)} f(v,u) + \sum_{v \in V_1(u) \cup V_2(u)} f'(u,v) - \sum_{v \in V_1(u) \cup V_2(u)} f'(v,u) \, . \qquad (26.6)
\end{aligned}
$$

In equation (26.6), we can extend all four summations to sum over $V$, since each additional term has value 0. (Exercise 26.2-1 asks you to prove this formally.) With all four summations over $V$, instead of just subsets of $V$, we get equation (26.5).

Now we are ready to prove flow conservation for $f \uparrow f'$ and that $|f \uparrow f'| = |f| + |f'|$. For the latter property, let $u = s$ in equation (26.5). Then, we have

$$
\begin{aligned}
|f \uparrow f'| &= \sum_{v \in V} (f \uparrow f')(s,v) - \sum_{v \in V} (f \uparrow f')(v,s) \\
&= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{v \in V} f'(s,v) - \sum_{v \in V} f'(v,s) \\
&= |f| + |f'| \, . \qquad (26.7)
\end{aligned}
$$

For flow conservation, observe that for any vertex $u$ that is neither $s$ nor $t$, flow conservation for $f$ and $f'$ means that the right-hand side of equation (26.5) is 0, and thus $\sum_{v \in V} (f \uparrow f')(u,v) = \sum_{v \in V} (f \uparrow f')(v,u)$. ■

### Augmenting paths

Given a flow network $G = (V, E)$ and a flow $f$, an ***augmenting path*** $p$ is a simple path from $s$ to $t$ in the residual network $G_f$. By the definition of the residual network, we may increase the flow on an edge $(u,v)$ of an augmenting path by up to $c_f(u,v)$ without violating the capacity constraint on whichever of $(u,v)$ and $(v,u)$ is in the original flow network $G$.

The shaded path in Figure 26.4(b) is an augmenting path. Treating the residual network $G_f$ in the figure as a flow network, we can increase the flow through each edge of this path by up to 4 units without violating a capacity constraint, since the smallest residual capacity on this path is $c_f(v_2, v_3) = 4$. We call the maximum amount by which we can increase the flow on each edge in an augmenting path $p$ the ***residual capacity*** of $p$, given by

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\} \ .$$

The following lemma, whose proof we leave as Exercise 26.2-7, makes the above argument more precise.

***Lemma 26.2***
Let $G = (V, E)$ be a flow network, let $f$ be a flow in $G$, and let $p$ be an augmenting path in $G_f$. Define a function $f_p : V \times V \to \mathbb{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p \ , \\ 0 & \text{otherwise} \ . \end{cases} \quad (26.8)$$

Then, $f_p$ is a flow in $G_f$ with value $|f_p| = c_f(p) > 0$. ■

The following corollary shows that if we augment $f$ by $f_p$, we get another flow in $G$ whose value is closer to the maximum. Figure 26.4(c) shows the result of augmenting the flow $f$ from Figure 26.4(a) by the flow $f_p$ in Figure 26.4(b), and Figure 26.4(d) shows the ensuing residual network.

***Corollary 26.3***
Let $G = (V, E)$ be a flow network, let $f$ be a flow in $G$, and let $p$ be an augmenting path in $G_f$. Let $f_p$ be defined as in equation (26.8), and suppose that we augment $f$ by $f_p$. Then the function $f \uparrow f_p$ is a flow in $G$ with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.

***Proof*** Immediate from Lemmas 26.1 and 26.2. ■

## Cuts of flow networks

The Ford-Fulkerson method repeatedly augments the flow along augmenting paths until it has found a maximum flow. How do we know that when the algorithm terminates, we have actually found a maximum flow? The max-flow min-cut theorem, which we shall prove shortly, tells us that a flow is maximum if and only if its residual network contains no augmenting path. To prove this theorem, though, we must first explore the notion of a cut of a flow network.

A ***cut*** $(S, T)$ of flow network $G = (V, E)$ is a partition of $V$ into $S$ and $T = V - S$ such that $s \in S$ and $t \in T$. (This definition is similar to the definition of "cut" that we used for minimum spanning trees in Chapter 23, except that here we are cutting a directed graph rather than an undirected graph, and we insist that $s \in S$ and $t \in T$.) If $f$ is a flow, then the ***net flow*** $f(S, T)$ across the cut $(S, T)$ is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \ . \quad (26.9)$$