

# Low Dimensions Counter-Examples to the Hirsch Conjecture

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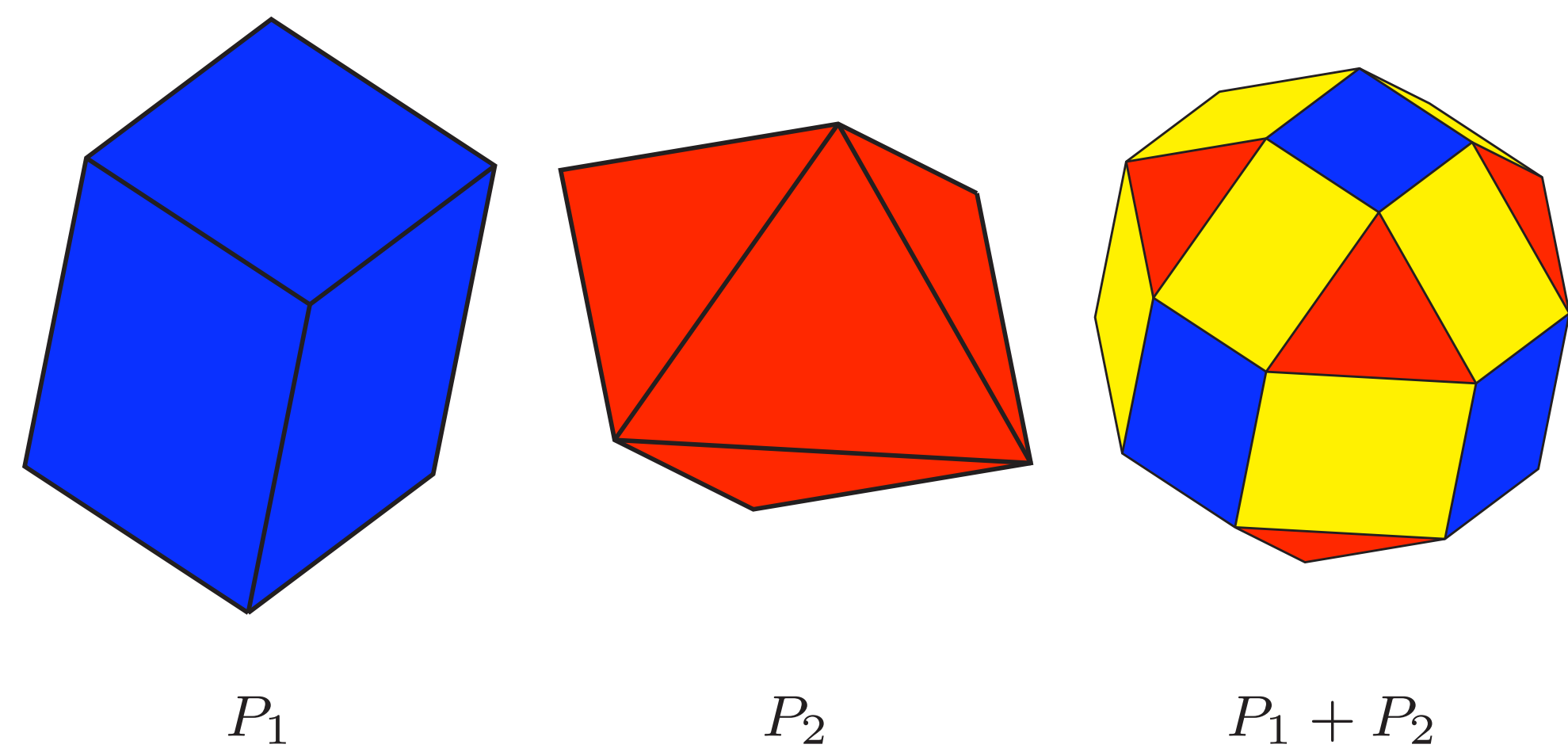
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## Contribution

The size of counter-examples to the Hirsch conjecture is exponential in their dimensions. The original counter-example, which has 43 dimensions, is so large that it cannot be computed in reasonable time, as it is estimated to have around  $2^{40}$  vertices. We show new counter-examples in dimensions as low as 20, with much lower number of vertices.

## Minkowski sums

For any two polytopes  $P_1$  and  $P_2$ , their Minkowski sum is defined by  $P_1 + P_2 = \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}$ . A property of Minkowski sums is that, when no facets of  $P_1$  and  $P_2$  are parallel, any facet of  $P_1$  or  $P_2$  results in a parallel facet of the same size and shape in  $P_1 + P_2$ . Such facets are called *pure facets*.



In this example, blue facets come from  $P_1$  and red facets come from  $P_2$ . Note that each blue facet shares vertices with red facets.

Counter-examples to the Hirsch conjecture can be built from any Minkowski sum  $P_1 + P_2$  such that two pure facets coming from different polytopes never share a vertex. This can only happen when the dimension  $d$  of  $P_1$  and  $P_2$  is at least 4.

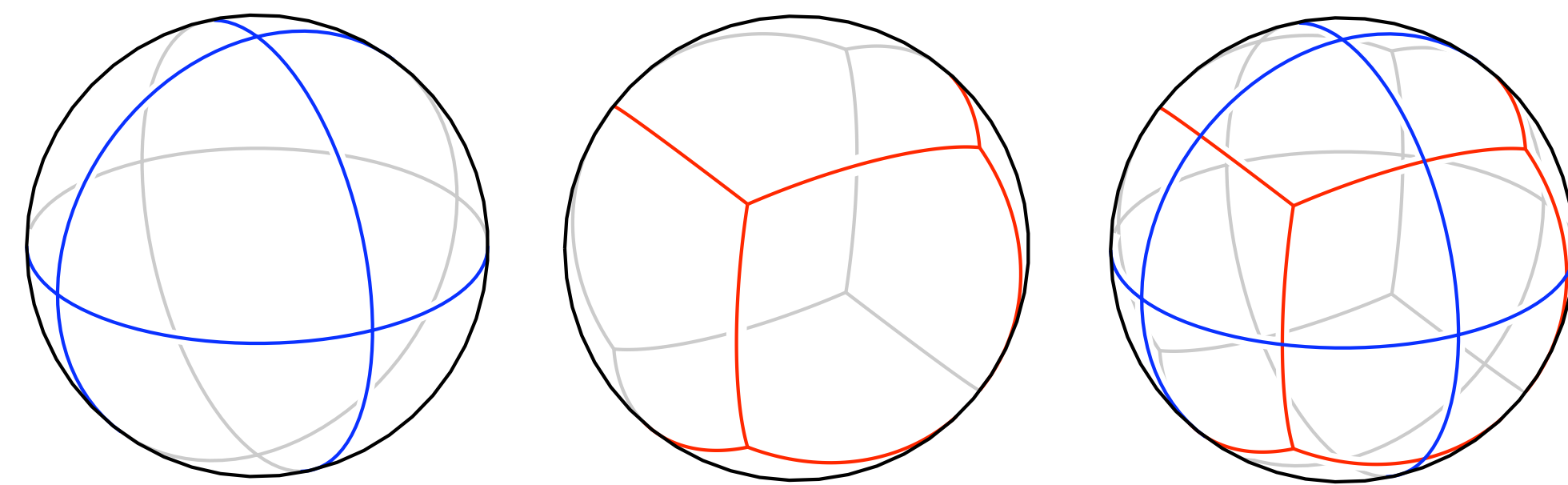
Critically, the dimension of the final counter-example is  $f_0(P_1) + f_0(P_2) - d - 1$ , where  $f_0(\cdot)$  is the number of vertices of a polytope. We therefore search for Minkowski sums satisfying the condition with a minimum number of vertices.

## References

[1] Francisco Santos A counter-example to the Hirsch conjecture preprint, June 2010, 27 pages.

## Normal maps

For any  $d$ -dimensional polytope  $P$  containing the origin, its *normal map*  $\mathcal{M}(P)$  can be defined as the central projection of its polar  $P^*$  from the origin on the unit sphere  $S^{d-1}$ . The projection defines a polyhedral cell complex on  $S^{d-1}$  whose combinatorial properties are similar to those of  $P^*$ .



$\mathcal{M}(P_1)$

$\mathcal{M}(P_2)$

$\mathcal{M}(P_1 + P_2)$

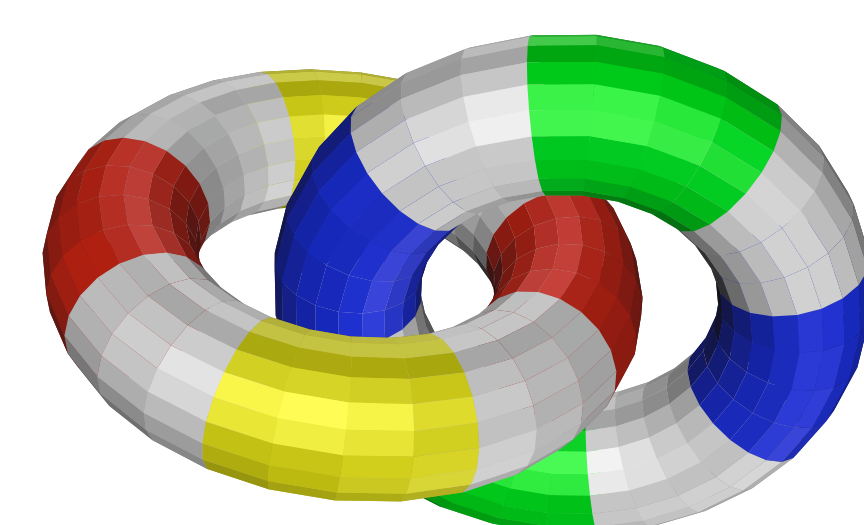
The normal map of a sum  $P_1 + P_2$  is the common refinement of the normal maps of  $P_1$  and  $P_2$ .

For the Minkowski sum to generate a counter-example, no cell of  $\mathcal{M}(P_1 + P_2)$  can have both a vertex originally in  $\mathcal{M}(P_1)$  and a vertex originally in  $\mathcal{M}(P_2)$ . Another way to say this is that for no pair of cells  $C_1$  of  $\mathcal{M}(P_1)$  and  $C_2$  of  $\mathcal{M}(P_2)$  can the cells each contain a vertex from the other one.

## Geometry of counter-examples

In all our counter-examples, as in the original one, the vertices of the normal maps  $\mathcal{M}(P_1)$  and  $\mathcal{M}(P_2)$  are contained in two doughnut-like shapes  $D_1$  and  $D_2$  respectively. The two doughnuts are subsets of  $S^3$ , and are imbricated one into the other, as illustrated below.

More precisely, the vertices of  $\mathcal{M}(P_1)$  are contained in the red and yellow regions, and the vertices of  $\mathcal{M}(P_2)$  are contained in the blue and green regions. If we denote as  $R, Y, G$  and  $B$  the red, yellow, green and blue regions respectively, and we denote by an arrow  $X \rightarrow Y$  the relation “ $X$  is contained in a cell whose vertices are in  $Y$ ”, then we obtain the diagram on the right.



$R \rightarrow G$

$\uparrow \quad \downarrow$

$B \leftarrow Y$

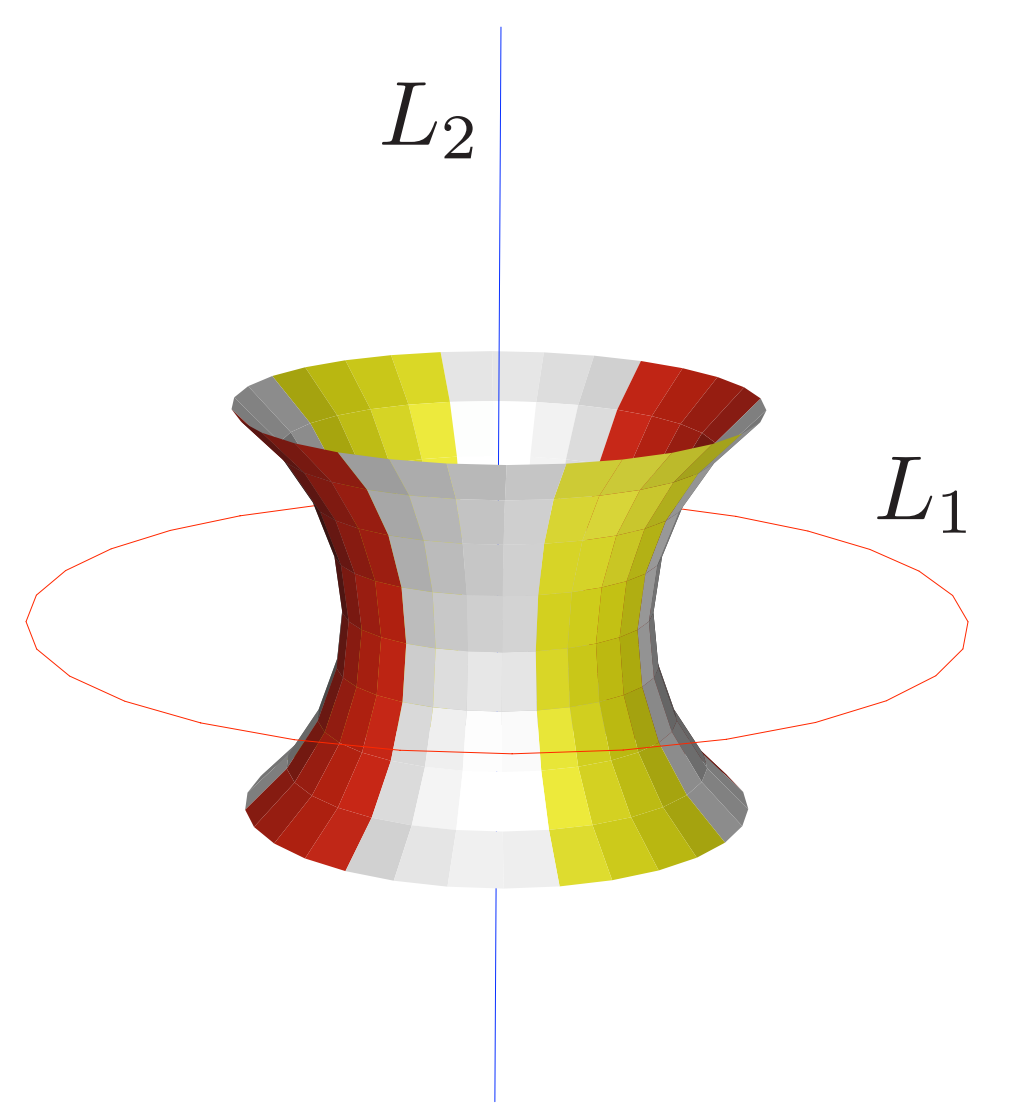
## 2D diagram

In order to keep up with the constraints on the position of vertices, Santos developed a 2D diagram based on the following loop interlocked in  $S^3$ :

$$L_1 = \{(x_1, x_2, 0, 0) : x_1^2 + x_2^2 = 1\} \quad L_2 = \{(0, 0, x_3, x_4) : x_3^2 + x_4^2 = 1\}$$

We project any point in  $S^3 - L_1 - L_2$  on a torus defined by  $x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2$ , using the unique geodesic connecting  $L_1$  to  $L_2$  through that point.

This gives a 2-dimensional toroidal map of  $S^3$ , on which we can easily check the conditions on the normal maps. The  $x$  coordinate indicates in which region of the doughnut  $D_1$  a node of  $\mathcal{M}(P_1)$  is, and which region of  $D_1$  is contained in which cell of  $\mathcal{M}(P_2)$ . The  $y$  coordinate gives the same information for  $D_2$ .

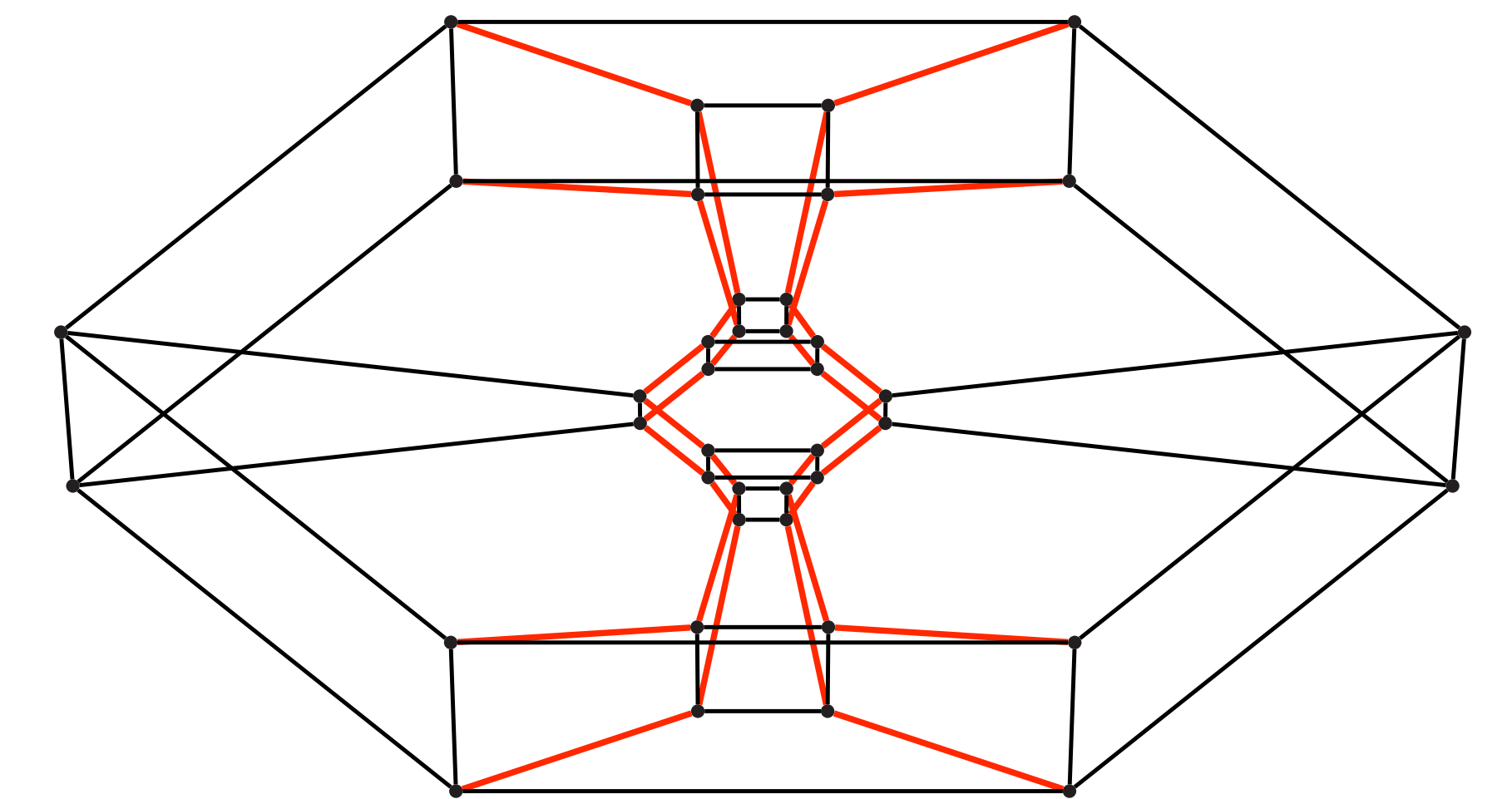


## Counter-examples

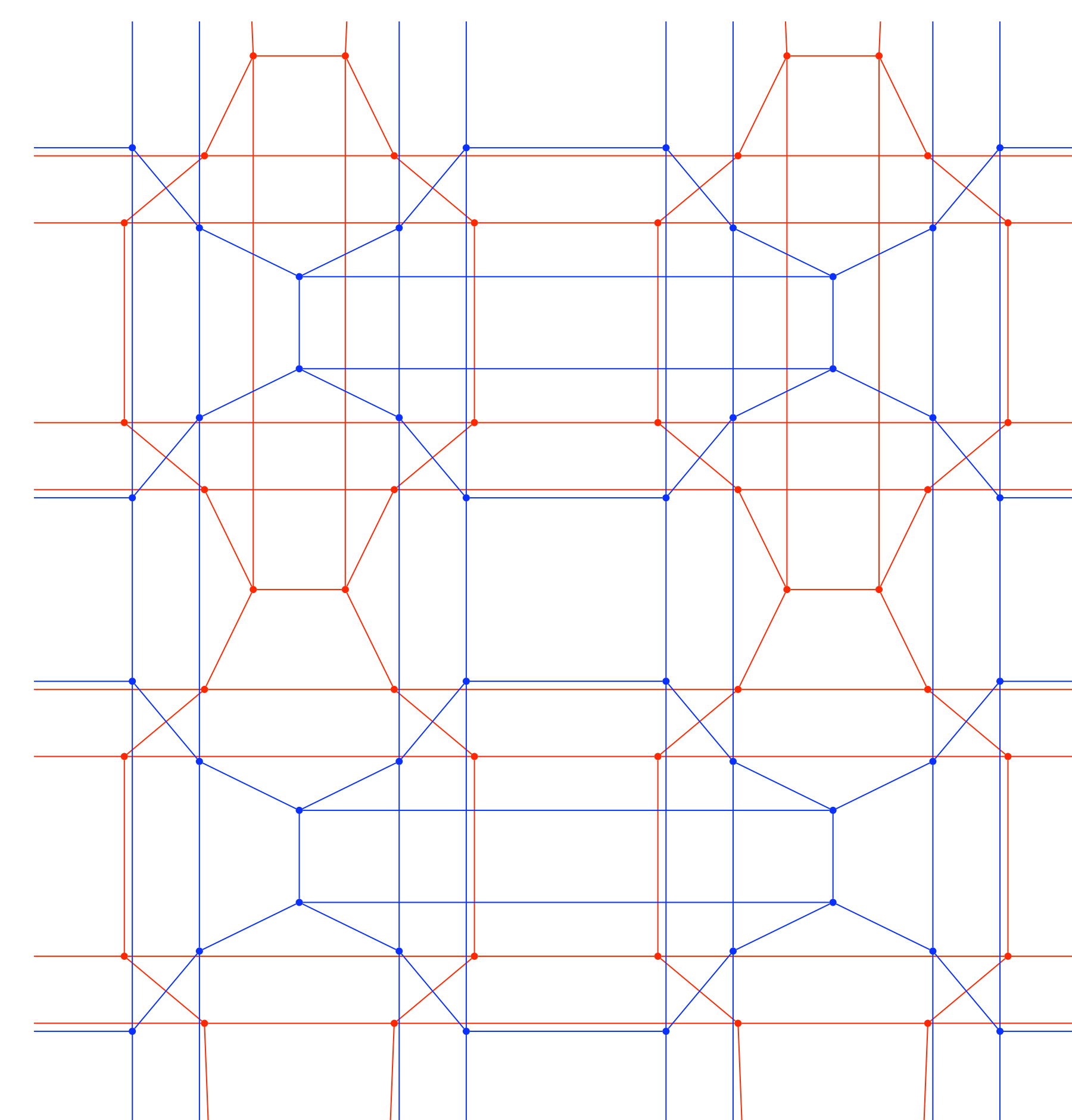
The original counter-example, in 43 dimensions, was based on two copies of a 24-vertices polytope.

Our first counter-example is in 23 dimensions, and is based on two copies of a 14-vertices polytope. The Schlegel diagram of its dual is to the right, indicating the aspect of its normal map, and the 2D diagram of the sum is below.

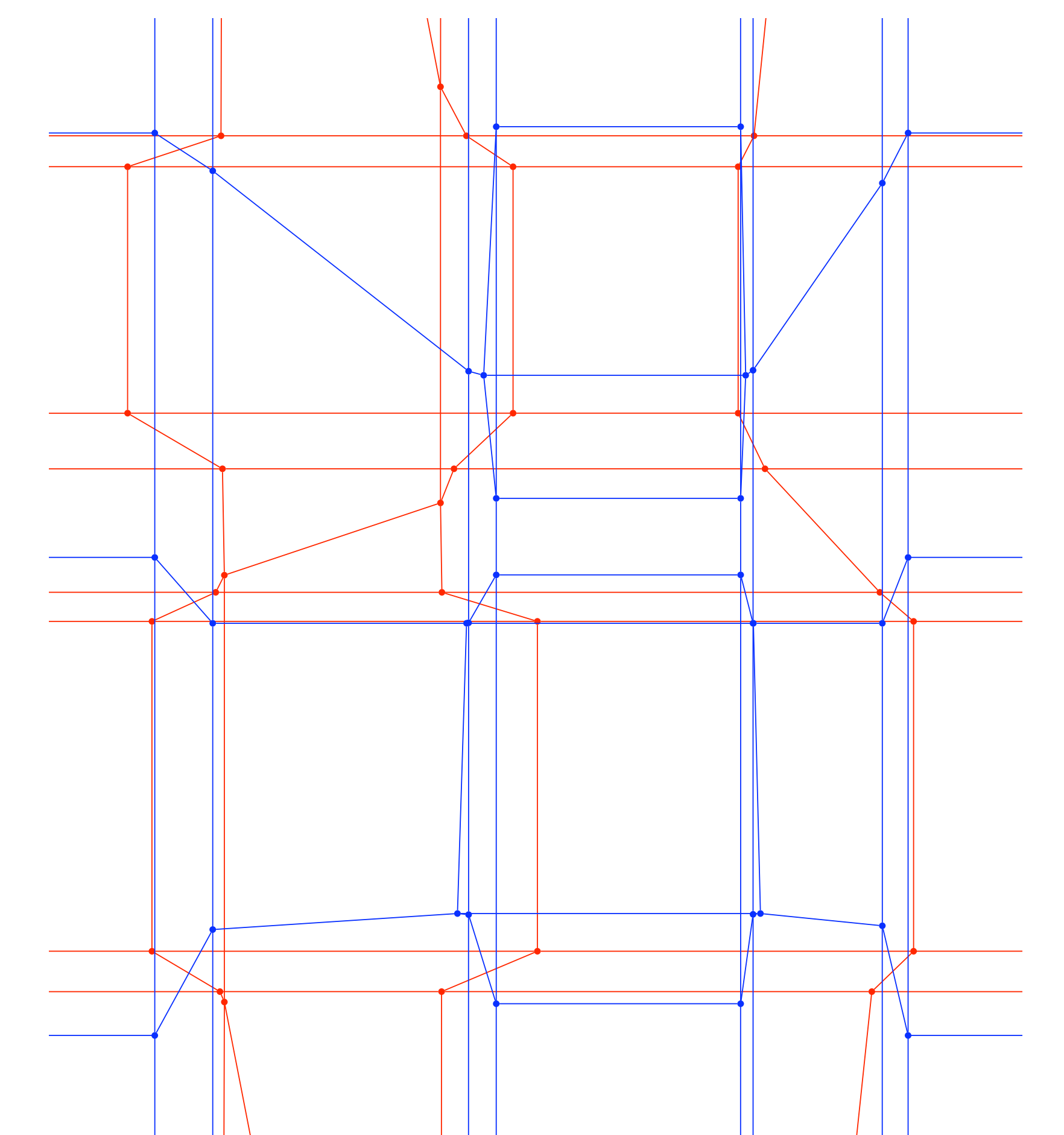
The second counter-example is in 20 dimensions, and is based on two polytopes of 13 and 12 vertices, obtained by removing one and two vertices from the previous polytope, and adjusting the position of others.



Schlegel diagram of the polar of a polytope of 14 vertices



Polytopes of 14 vertices  
Counter-example in 23 dimensions



Polytopes of 12 and 13 vertices  
Counter-example in 20 dimensions