Homework 3
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Amit Chakrabarti
Department of Computer Science
Dartmouth College

As usual, please think carefully about how you are going to organise your proofs *before* you begin writing. Make sure each solution fits within a page, as per the homework guidelines (Handout 2).

1. As we remarked in class, it is clear that monotone circuits can only compute monotone functions. Prove the converse, i.e., prove that any n-bit monotone Boolean function can be computed by an n-input monotone circuit.

2. Let p_n be the parity function on n variables. Consider depth-2 circuits which are given the input Boolean vector $\vec{x}$ both in unnegated and negated form. As part of our proof that $p_n \notin \mathrm{AC}^0$ we showed that if such a circuit computes p_n, it must have size at least 2^{n-1}. But what if we're only interested in a circuit which computes p_n correctly for a little more than half of the 2^n different inputs?

   2.1. Why is it not interesting to compute p_n correctly on just 2^{n-1} inputs?

   2.2. Show that there is a depth-2 circuit of size $2^{O(\sqrt{n})}$ that computes p_n correctly on at least $2^{n-1} + 2^{\sqrt{n}}$ inputs.

3. We proved in class that the parity function is not in $\mathrm{AC}^0[3]$, a supposedly stronger result than its not being in AC^0. Prove that it *is* in fact a stronger result by showing that $\mathrm{AC}^0[3]$ is a *proper* superset of AC^0. In other words, exhibit a function in $\mathrm{AC}^0[3]$ that is not in AC^0.

   Hint: You can reuse large parts of the random restrictions proof. You don't have to write proofs for things we did in class already.

4. The n-bit majority function takes n Boolean inputs and outputs 1 iff at least $n/2$ of the inputs are 1. Prove that this function is not in AC^0.

   Hint: I know two ways to do this. You can do it directly, mimicking the proof we gave in class for parity. Or you can give a shorter solution by exhibiting an AC^0 circuit which reduces parity to majority. If using the latter approach, it might help to use $\textsc{false} = +1$, $\textsc{true} = -1$ and consider sums of the form $x_1 + \cdots + x_{n/2} - x_{n/2+1} - \cdots - x_n$. Be careful about separating the two cases: (a) n is odd (b) n is even.