# 1 What is a Linear Programming Problem?

A *linear program* (LP) is a minimization problem where we are asked to minimize a given linear function subject to one or more linear inequality constraints. The linear function is also called the *objective function*.
Formulation:

$$Minimize \sum_{i=1}^{n} C_i X_i \qquad \text{(where } C_i \in \Re \text{ and are constants and } X_i \in \Re \text{ and are variables)}$$

Subject to constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &\geq b_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n &\geq b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n &\geq b_n \end{aligned}$$

Alternately, we can rewrite the above formulation as:

$$\text{Minimize } C^T X \qquad \text{(where } C, X \in \Re \text{ and are column vectors)}$$

Subject to constraints:

$$AX \geq b \qquad \text{(where } b \in \Re^m, A \in \Re^{m*n}\text{)}$$

Given $C, A$ and $b$ the above LP can be solved in time $poly(inputlength)$

# 2 Vertex Cover

*Vertex Cover:* A given subset of vertices of a graph G that covers all the edges in G. For every edge $(u, v)$ in the original graph, either vertices $u$ or $v$ or both are in the vertex cover. *Note:* If total number of vertices is n, there are $2^n$ possible subsets.

A solution to a general LP gives: a sequence of real numbers $x_1, x_2 \ldots, x_n$
Suppose for a moment that all $x_i$ are in the range { 0, 1 }
*Note:* If all $x_i$ are in the range { 0, 1 }, there are $2^n$ possible assignments for $x_1, x_2 \ldots, x_n$

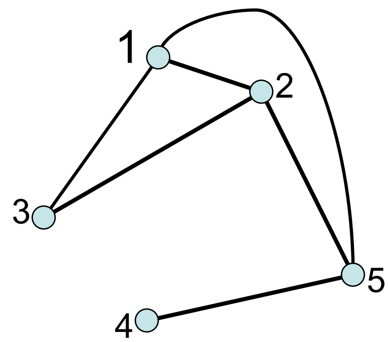
So let us assign a binary value to variable $x_i$ to vertex $i$.

$$x_i = \begin{cases} 1 & i \in Subset \\ 0 & i \notin Subset \end{cases}$$

## 2.1 Example:

From the first figure (Figure 1):
$(x_1, x_2, x_3, x_4, x_5, x_6) =$ (0,1,1,0,0,1) is not a cover
but $(x_1, x_2, x_3, x_4, x_5, x_6) =$ (0,1,0,1,0,1) is a cover

Figure 1: Example graph for vertex cover calculation




## 2.2 Vertex Cover Formulation

Let us require that $C \subseteq V$ be a vertex cover.
$\equiv$ to requiring that $\forall (i,j) \in E$ that either $i \in C$ or $j \in C$

$\equiv$ to requiring that $\forall (i,j) \in E$ , $x_i + x_j \geq 1$
*Reformulation of problem:* Vertex Cover problem can be written as

$$\text{Minimize } x_1 + x_2 + x_3 + \ldots + x_n$$

Subject to constraints:

$$x_i + x_j \geq 1 \qquad \text{(for each edge } (i,j) \in E)$$

$$x_i \in \{0,1\} \qquad \text{(for each vertex } i)$$

*Note:* Above problem is not an LP since above statement is not a linear constraint. The above problem is actually an Integer Linear Problem or IP. Solving an IP is NP-Complete.

Let us now relax our second constraint to $0 \leq x_i \leq 1$ for each i and allow $x_i \in \Re$.

Here we are violating our original inequality direction, since $x_i \leq 1$
We can easily fix this problem by restating our constraint as:

$$x_i \geq 0 \qquad \text{(for each vertex } i)$$

$$-x_i \geq -1 \qquad \text{(for each vertex } i)$$

**Example**

$$\text{Minimize } X_1 + X_2 + X_3 + X_4 + X_5$$

Subject to constraints:

$$
\begin{aligned}
X_1 + X_2 &\geq 1 \\
X_1 + X_3 &\geq 1 \\
X_2 + X_3 &\geq 1 \\
X_2 + X_4 &\geq 1 \\
X_4 + X_5 &\geq 1 \\
&\vdots \\
X_1 &\geq 0 \\
-X_1 &\geq -1 \\
X_2 &\geq 0 \\
-X_1 &\geq -1 \\
&\vdots \\
-X_5 &\geq 0 \\
-X_5 &\geq -1
\end{aligned}
$$

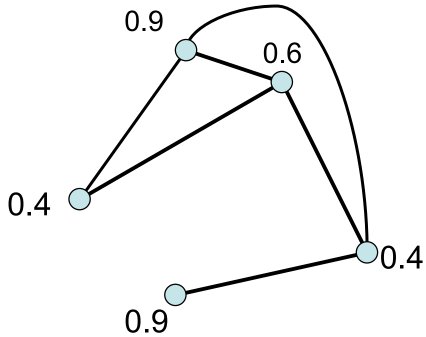
This can be rewritten in matrix form as:

$$Y \times X \geq Z$$

where

$$
Y = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
 & & \ddots & & \\
 & & & \ddots & \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

Figure 2: Possible real valued LP solution for constraints on $X_1..X_5$ in our example




$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 t \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \\ 0 \\ -1 \\ 0 \\ -1 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

## Proposed Algorithm: LP Rounding Algorithm for Vertex Cover

**Algorithm 1**: VertexCover$(V, E)$

Construct LP relaxation for given instance $(V, E)$
Invoke polynomial time LP solver to get a vector $X^* \in \Re^n$ that minimizes $\sum_{i=1}^n x_i$
$C \longleftarrow \emptyset$
**for** i = 1 to n **do**
  if $x_i^* \geq 1/2$ Then $C \longleftarrow C \cup \{i\}$;
**return** $C$

Let us now verify that the above algorithm is correct and analyze its optimality.

***Claim 1:*** *Returned set $C$ of vertices is a Vertex Cover*
We know from our constraints that $\forall (i,j) \in E$ that $x_i^* + x_j^* \geq 1$. Therefore at least one of $x_i^*$ or $x_j^* \geq 1/2$ and so at least one of the vertices $i$, $j$ from the edge $(i,j)$ must $\in C$. *Hence the claim is proved.*

***Let our Cost function be:*** $Cost(X) = |X|$

***Claim 2:*** *If $C^*$ is the min cost vertex cover then the cost of* $(C) \leq 2 * cost(C^*)$ In other words the LP rounding algorithm is a 2-approximation.

***Proof:***
Let $Z^* = x_1^* + x_2^* + x_3^* + \ldots + x_n^*$

$Z^*$ is the "$Cost$" of the LP's optimal solution. ( This is the sum of real numbers and not the size of any set)

Since $X^*$ is optimal for the LP:

$$Z^* \leq Cost(C^*) \tag{1}$$

The binary solution $\tilde{x}$ obtained from $C^*$ i.e Set variables in $\tilde{x}_i$ as 0 or 1 depending on the optimal solution.

$$\begin{aligned} \sum_{i=1}^{n} x_i^* &\leq \sum_{i=1}^{n} \tilde{x}_i \\ \sum Optimum_{LP} &\leq \sum Optimum_{IP} \end{aligned}$$

Let $x = (x_1, x_2, x_3, \ldots, x_n)$ be the IP solution implicitly produced by the algorithm.

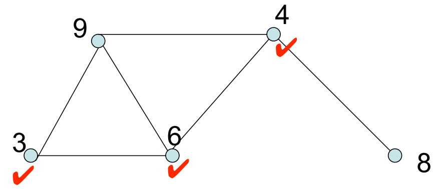
$$\begin{aligned} x_i &\leq 2x_i^* \qquad \ldots \forall i \\ \Longrightarrow \sum_{i=1}^{n} x_i &\leq 2 \sum_{i=1}^{n} x_i^* \qquad (2) \\ \Longrightarrow Cost(C) &\leq 2Z^* \qquad (3) \end{aligned}$$

From inequalities (1) and (3) we get

$$\Longrightarrow Cost(C) \leq 2 \times Cost(C^*)$$

*Thus proved that the LP rounding algorithm is a 2-approximation .*

Figure 3: Possible solution considering min cost of vertex cover




## 2.3 Vertex Cover considering cost of a vertex

This technique allows us to incorporate the idea of cost of a vertex into our model and find a vertex cover of minimum total cost.
Inequality (2) would need to be changed from

$$\sum_{i=1}^{n} x_i \leq 2 \sum_{i=1}^{n} x_i^*$$

to

$$\sum_{i=1}^{n} c_i x_i \leq 2 \sum_{i=1}^{n} c_i x_i^*$$

Here in our example in Figure 3 our cost = 3 + 6 + 4 = 13 is minimum possible of costs of all minimum vertex covers.