1. For a complexity class $C$, define two new complexity classes $\exists C$ and $\forall C$ as follows.

$$\exists C = \{ x \in \Sigma^* : \exists y \in \Sigma^* \text{ with } |y| = \text{poly}(|x|) \text{ such that } \langle x, y \rangle \in L_0 : L_0 \in C \}$$

$$\forall C = \{ x \in \Sigma^* : \forall y \in \Sigma^* \text{ with } |y| = \text{poly}(|x|) \text{ we have } \langle x, y \rangle \in L_0 : L_0 \in C \}$$

The notation $|y| = \text{poly}(|x|)$ means that there is some fixed polynomial $p$ such that $|y| = O(p(|x|))$.

Prove, rigorously, that $\exists P = \text{NP}$ and $\forall P = \text{coNP}$. Use only the basic definitions from class, where $\text{NP}$ was defined using NDTMs and $\text{coNP}$ was defined as $\{ L \subseteq \Sigma^* : \overline{L} \in \text{NP} \}$. (This problem is simply asking you to write out a rigorous version of ideas we have already discussed in class.) \[2 \text{ points}\]

2. Define the following two complexity classes

$$\text{EXPTIME} = \bigcup_{i=1}^{\infty} \text{DTIME}(2^{n^i})$$

$$\text{NEXPTIME} = \bigcup_{i=1}^{\infty} \text{NTIME}(2^{n^i})$$

Prove that $P = \text{NP}$ implies $\text{EXPTIME} = \text{NEXPTIME}$. The key trick is to “pad” an input with a lot of extra symbols. \[2 \text{ points}\]

3. For this problem, assume that Boolean formulas are encoded as strings over the alphabet $\{0, 1, \lor, \land, \neg, (, )\}$, fully parenthesized to resolve ambiguities. Note that the negation operator ($\neg$) has higher priority than the other two. The variables in a formula $\phi$ are represented as binary substrings of $\phi$ with no leading zeros. For instance, the formula

$$(x_1 \land \neg x_2) \lor \neg x_1 \lor (x_3 \land \neg x_4) \land \neg x_3 \lor x_5$$

is represented as the string

$$(1 \land \neg 10) \lor \neg 1 \lor (11 \land \neg 1) \land \neg 100 \land (\neg 11 \lor 101).$$

Define $\text{SATISFIES} = \{ \langle \phi, \alpha \rangle : \phi \text{ is a Boolean formula and the assignment } \alpha \text{ satisfies } \phi \}$. Our proof that $\text{SAT} \in \text{NP}$ boiled down to showing that $\text{SATISFIES} \in \text{P}$. Prove the stronger result that $\text{SATISFIES} \in \text{L}$ (i.e., LOGSPACE). Recall, from class, that $L \subseteq \text{P}$.

This problem is all about careful implementation, so take care to specify exactly how you use the work tape of your TM. Some naive implementations end up requiring $\Omega(\log^2 n)$ space. \[2 \text{ points}\]