If you have any questions about the grading of a particular problem, please consult the appropriate grader as shown below:

- Problems 1 and 2 were graded by David Blinn.
- Problem 3 was graded by Chien-Chung Huang.
- Problems 4 and 5 were graded by Amit Chakrabarti.

1. Let \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \) be the set of all non-negative integers and let \( \mathbb{R}^+ \) be the set of all non-negative real numbers. Suppose we have two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). Write precise mathematical definitions of the notations \( f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \).

**Solution:** \( f(n) = O(g(n)) \) means \( \exists c, n_0 > 0 (\forall n \geq n_0 (f(n) \leq cg(n))). \)
\( f(n) = \Omega(g(n)) \) means \( \exists c, n_0 > 0 (\forall n \geq n_0 (f(n) \geq cg(n))). \)

2. Let \( S \) be a sample space for a random process and let \( P \) be the appropriate probability distribution on \( S \). Answer the following questions with precise mathematical definitions.

2.1. What does it mean to say that \( P \) is a uniform probability distribution?

**Solution:** Any of the following would be an appropriate definition:
\[ \forall x \in S \ (P(x) = 1/|S|) \]
OR
\[ \forall x, y \in S \ (P(x) = P(y)) \]
OR
\( P \) assigns equal weight to all elements of \( S \).

2.2. Let \( A, B \subseteq S \) be two events. What does it mean to say that \( A \) and \( B \) are independent?

**Solution:** It means \( P(A \mid B) = P(A) \).
3. For each of the recurrences below, find a big-$\Theta$ bound on the solution. You may use theorem(s) and result(s) from the textbook without proof. If you ever need to use the fact that $f(n) = \Theta(g(n))$ for some $f$ and $g$, you should state this, but you need not prove it.

3.1. 

$$T(n) = \begin{cases} 4T(\lceil n/2 \rceil) + 2n^2 - 9n + 6, & \text{if } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

**Solution:** We have $2n^2 - 9n + 6 = \Theta(n^2)$. We also have $\log_2 4 = 2$. The master theorem tells us that $T(n) = \Theta(n^2 \log n)$.

3.2. 

$$T(n) = \begin{cases} 27T(\lceil n/3 \rceil) + n^3, & \text{if } n > 1 \\ 2, & \text{if } n = 1. \end{cases}$$

**Solution:** The initial condition $T(1) = 2$ doesn’t matter; we have $\log_3 27 = 3$ and so the master theorem tells us that $T(n) = \Theta(n^3 \log n)$.

3.3. 

$$T(n) = \begin{cases} 4T(\lceil n/3 \rceil) + \frac{n^2}{\sqrt{n+1}}, & \text{if } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

**Solution:** We have $\frac{n^2}{\sqrt{n+1}} = \Theta(n^{3/2})$, so we want to compare $\log_4 4$ with $3/2$. Since $3^{3/2} = \sqrt{27} > \sqrt{16} = 4$, we have $3/2 > \log_4 4$. Now the master theorem tells us that $T(n) = \Theta(n^{3/2})$. 


3.4. \[ T(n) = \begin{cases} 4T([n/2]) + n \log n, & \text{if } n > 1 \\ 2, & \text{if } n = 1. \end{cases} \]

**Solution:** Let us define two new sequences \( L(n) \) and \( U(n) \) as follows: \( L(1) = U(1) = 2 \), and for \( n > 1 \),

\[
L(n) = 4L([n/2]) + n, \\
U(n) = 4U([n/2]) + n^{3/2}.
\]

Then, for all \( n \), \( L(n) \leq T(n) \leq U(n) \) (this can be proved rigorously by induction). But the master theorem tells us that \( L(n) = \Theta(n^2) \) and \( U(n) = \Theta(n^2) \). Therefore \( T(n) = \Theta(n^2) \) as well.

This problem was almost identical to textbook P4.5-7 from the textbook. The hint at the back of the textbook suggests a different, but more long-winded, approach. (Although, why on earth would one want to work harder than necessary?!) 

4. As discussed in class, a virtual 3-sided die that shows the numbers \( x_1, x_2 \) and \( x_3 \) with equal probability can be constructed out of a usual 6-sided die by writing each of the \( x_i \) on two of the faces. Suppose that we construct three specific 3-sided dice as follows:

- The red die shows the numbers in \( \{3, 5, 7\} \) with equal probability.
- The green die shows the numbers in \( \{2, 4, 9\} \) with equal probability.
- The blue die shows the numbers in \( \{1, 6, 8\} \) with equal probability.

We say that “die \( A \) beats die \( B \)” if, upon rolling both dice, the number on top of \( A \) exceeds the number on top of \( B \) with probability \( > 50\% \).

4.1. Suppose the red die and the green die are both rolled. Let \( E_{RG} \) denote the event that the number on top of the red die exceeds the number on top of the green die. Compute \( P(E_{RG}) \). Does the red die beat the green die?

**Solution:** A sample space for rolling these two dice is given by

\[ S_{RG} = \{ (i, j) : \text{the red die shows } i \text{ and the green die shows } j \}. \]

Since the dice are fair, the probability distribution on this sample space is uniform. We see that \( |S_{RG}| = 9 \) and that \( E_{RG} = \{ (3, 2), (5, 2), (5, 4), (7, 2), (7, 4) \} \), so that \( |E_{RG}| = 5 \). Therefore, using the formula for the probability of an event under a uniform distribution, we have

\[ P(E_{RG}) = \frac{|E_{RG}|}{|S_{RG}|} = \frac{5}{9} > 50\%, \]

which means that the red die does beat the green die.
4.2. Does the green die beat the blue die? Show your work!

Solution: Proceeding as above, the situation is modelled by a uniform distribution on the sample space

\[ S_{GB} = \{(i, j) : \text{the green die shows } i \text{ and the blue die shows } j\} \]

and the event \( E_{GB} = \{(2, 1), (4, 1), (9, 1), (9, 6), (9, 8)\} \). Therefore

\[ P(E_{GB}) = \frac{|E_{GB}|}{|S_{GB}|} = \frac{5}{9} > 50\% \]

and we conclude that the green die does beat the blue die.

4.3. Does the blue die beat the red die? Again, show your work.

Solution: A similar calculation gives a probability of \( 5/9 \) again and shows that (wonder of wonders) the blue die does beat the red die! Yet another way in which probability can be non-intuitive.
5. A standard deck of 52 cards is randomly shuffled and separated into four bridge hands (traditionally called North, East, South and West) of 13 cards each. A little birdie tells you that North and South hold 11 spades between them, and thus, East and West hold 2 spades between them. Given this information, what is the probability that East and West hold exactly one spade each?

Free hint: The answer is not 50%.

Solution: Let $S$ be the set of cards held by East and West. It does not matter exactly which cards are in $S$, just that $|S| = 26$ and $S$ contains exactly two spades. Now, we divide $S$ into two hands of 13 cards each. The given question is: what is the probability that both East and West end up with one spade each?

There are $\binom{26}{13}$ ways to divide $S$ into two hands, all equally likely. Of these, let us count the number of divisions in which East gets exactly one spade. We can choose which spade (s)he gets in $\binom{2}{1}$ ways and which 12 non-spades (s)he gets in $\binom{24}{12}$ ways; by the product principle, this gives a total of $\binom{2}{1}\binom{24}{12}$ ways. Therefore,

$$
\text{Required probability} = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}
= \frac{2 \cdot 24!}{12! \cdot 13!}
= \frac{2 \cdot 24! \cdot 13! \cdot 13!}{26! \cdot 12! \cdot 12!}
= \frac{2 \cdot 13 \cdot 13}{26 \cdot 25}
= \frac{13}{25}.
$$

Thus, the desired answer is $13/25 = 52\%$. 
