CS19: Solutions to Homework 1

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Throughout this homework, $\mathbb{Z}$ refers to the set of all integers.

1. **(Roster and Set-Builder Notations)** Here are some sets described in set-builder notation. Describe each of them in roster notation. You do not need to show any steps.

   (a) $\{x^2 : x \in \mathbb{Z} \text{ and } x^2 < 20\}$
   Solution: $\{0, 1, 4, 9, 16\}$

   (b) $\{k \in \mathbb{Z} : 10 \leq k \leq 99 \text{ and the sum of the digits of } k \text{ is } 9\}$
   Solution: $\{18, 27, 36, 45, 54, 63, 72, 81, 90\}$

   (c) $\{x \in \mathbb{Z} : 0 \leq x \leq 10 \text{ and } \frac{x}{2} \notin \mathbb{Z}\}$
   Solution: $\{1, 3, 5, 7, 9\}$

   (d) $\{S : S \subseteq \{a, b\}\}$
   Solution: $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$

   (e) $\{S : S \subseteq \{a, b, c, d\} \text{ and } |S| \text{ is even}\}$
   Solution: $\{\{\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}\}$

   (f) $\{S : \{1, 2\} \subseteq S \subseteq \{1, 2, 3, 4\}\}$
   Solution: $\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$

   (g) $\{S \subseteq \{1, 2, 3, 4\} : S \text{ is disjoint from } \{2, 3\}\}$
   Solution: $\{\{\}, \{1\}, \{4\}, \{1, 4\}\}$

2. **(Operations on Sets)** Let $A = \{1, 2, 3, 4, 5, 6\}, B = \{2, 4, 6, 8, 10\}$ and $C = \{0, 1, 5, 6, 9\}$. In the following subproblems, you must show your steps for those cases where the statement asks you to "verify" an equation. For the rest, you do not need to show any steps.

   (a) What is $A \cup B$? What is $(A \cup B) \cup C$?
   Solution:
   $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$. $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8, 10, 0, 9\}$

   (b) What is $B \cup C$? What is $A \cup (B \cup C)$?
   Solution:
   $B \cup C = \{2, 4, 6, 8, 10, 0, 1, 5, 9\}$. $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8, 10, 0, 9\}$

   (c) What is $A \cap B \cap C$?
   Solution:
   $A \cap B \cap C = \{6\}$

(d) Verify by direct computation that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Solution:
$(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6, 8, 10\} \cap C = \{1, 5, 6\}$
$(A \cap C) \cup (B \cap C) = \{1, 5, 6\} \cup \{6\} = \{1, 5, 6\}$

(e) What is $A - B$? What is $B - C$?
Solution:
$A - B = \{1, 3, 5\}$. $B - C = \{2, 4, 8, 10\}$

(f) What is $(A - B) - C$? What is $A - (B - C)$?
Solution:
$(A - B) - C = \{3\}$. $A - (B - C) = \{1, 3, 5, 6\}$

(g) Verify by direct computation that $(A - B) - C = A - (B \cup C)$.
Solution:
$(A - B) - C = \{3\}$
$A - (B \cup C) = A - \{2, 4, 6, 8, 10, 0, 1, 5, 9\} = \{3\}$

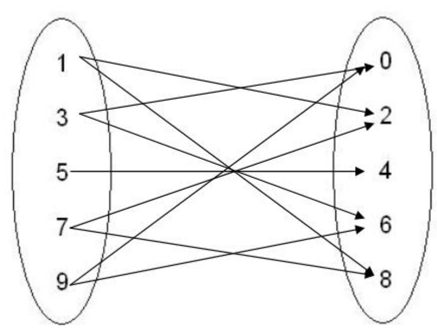
(h) Verify by direct computation that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.
Solution:
$A - (B - C) = A - \{2, 4, 8, 10\} = \{1, 3, 5, 6\}$
$(A - B) \cup (A \cap B \cap C) = \{1, 3, 5\} \cup \{6\} = \{1, 3, 5, 6\}$.

(i) What is $(A \cap B) \times (B - C)$?
Solution:
$(A \cap B) \times (B - C) = \{2, 4, 6\} \times \{2, 4, 8, 10\} =$
$\{(2, 2), (2, 4), (2, 8), (2, 10), (4, 2), (4, 4), (4, 8), (4, 10), (6, 2), (6, 4), (6, 8), (6, 10)\}$

3. **(Relations)** Let $S = \{1, 3, 5, 7, 9\}$ and $T = \{0, 2, 4, 6, 8\}$. Let's say that an element $x \in S$ "completes" an element $y \in T$ if $x + y$ is divisible by 3. Describe the relation "completes" from S to T as a subset of $S \times T$ (i.e., write out all the pairs in this relation). Then describe the same relation pictorially, using arrows, as done in class.

Solution:

$\{(1, 2), (1, 8), (3, 0), (3, 6), (5, 4), (7, 2), (7, 8), (9, 0), (9, 6)\}$



4. **(Special types of relations)** For each of the following relations, state whether or not it is (a) reflexive, (b) symmetric, (c) antisymmetric and (d) transitive. Whenever your answer is "no", explain why. This means that if, for instance, you say that a relation R is not symmetric, you must exhibit a pair (a, b) such that $(a, b) \in R$ but $(b, a) \notin R$.

(a) The relation "divides", on the set of all positive integers. (Note: we say that m divides n if n/m is an integer.)

Solution: reflexive: yes; symetric: no (1,2); antisymetric: yes; transitive: yes

(b) The relation "is disjoint from", on $\mathcal{P}(\mathbb{Z})$. (Note: $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$.)

Solution: reflexive: no $\{1\}$ is in $\mathcal{P}(\mathbb{Z})$, but ($\{1\}$,$\{1\}$) is not in our "disjoint from" relation; symetric: yes; antisymetric: no ($\{1\}$,$\{2\}$); transitive: no ($\{1,2\}$,$\{3,4\}$) and ($\{3,4\}$,$\{1,5\}$)

(c) $\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m > 0, n > 0$ and the sum of the digits of m equals the sum of the digits of $n\}$.

Solution: reflexive: yes; symetric: yes; antisymetric: no (1,10); transitive: yes

5. **(Functions)** Suppose $f : A \to B$ is a function. Define the relation f^{-1} from B to A as follows:

$$f^{-1} \;=\; \{(y, x) \in B \times A : \, f(x) = y\} \, .$$

(a) Suppose f^{-1} is also a function. What can you then conclude about f with regard to surjectivity and injectivity?

Solution: You can conclude that f is both surjective and injective. A function, by definition, has an output for every input, and exactly one output for every input. Because f^{-1} is also a function, its input (f's output) has an output (f's input), making f surjective. Also, its input (f's output) has exactly one output (f's input), making f injective.

(b) Does the implication work the other way, i.e., given that f has the properties that you came up with, above, does it follow that f^{-1} is a function? Briefly explain why or why not.

Solution: It does follow that f^{-1} is a function. Simply note that if f is surjective, every input to f^{-1} must have an output and if f is injective, f^{-1} must have only one output for every input, and thus f^{-1} is a function.

By the way, the notation f^{-1} is pronounced "f inverse".

6. **(Logic)** Let p and q denote arbitrary propositions.

(a) Write out a truth table for the compound proposition $\neg(p \Rightarrow q)$. Your table should have 4 columns: one for p, one for q, one for $p \Rightarrow q$ and finally one for your target $\neg(p \Rightarrow q)$.

Solution:

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$
T	**T**	**T**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**F**	**T**	**F**

(b) Write out a truth table for $(p \Rightarrow \neg q)$. Is $\neg(p \Rightarrow q)$ logically equivalent to $(p \Rightarrow \neg q)$?

Solution: No, $\neg(p \Rightarrow q)$ is not logically equivalent to $(p \Rightarrow \neg q)$. To see why, compare the final column of the truth table below with that of the solution to the problem above.

p	q	$\neg q$	$p \Rightarrow \neg q$
T	**T**	**F**	**F**
T	**F**	**T**	**T**
F	**T**	**F**	**T**
F	**F**	**T**	**T**

(c) Using a truth table, verify that $\neg(p \Rightarrow q)$ is logically equivalent to $(p \wedge \neg q)$. Does this make intuitive sense to you? Give an explanation in plain English why the opposite of the statement "p implies q" is the statement "p and not q".

Solution:

p	q	$\neg q$	$p \wedge \neg q$	$p \Rightarrow q$	$\neg(p \Rightarrow q)$
T	**T**	**F**	**F**	**T**	**F**
T	**F**	**T**	**T**	**F**	**T**
F	**T**	**F**	**F**	**T**	**F**
F	**F**	**T**	**F**	**T**	**F**

The opposite of "p implies q" is the same as the statement "p and not q" because "p implies q" is false only when p is true and q is false. Thus, its opposite is true when p is true and q is false. Notice that "p and not q"are true in the same circumstances.