CS19: Solutions to Homework 7

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1. By simple application of linearity of expectation, we get $n(\frac{1+2+3+4+5+6}{6}) = \frac{21n}{6} = \frac{7n}{2}$.

2. Suppose the students really knows the answer of x questions and guesses at the other y ones. How many questions can he answer correctly? Since there are five choices, so when he makes a guess, he has one-fifth chance. Thus, on the average, he has $x + y/5$ questions right and $(4/5)y$ questions wrong. So, what is the reasonable formula for computing his corrected score? A little thought tells us that

$$E(x + y/5) - cE((4/5)y) = E(x)$$

Obviously, c should be $1/4$. Put in words, it means we should subtract one-fourth of incorrect answers from correct ones.

3. We have to consider two cases.

   (a) $n \leq m$, then by definition of conditional probability, $P(X \geq n | X \geq m) = 1$.

   (b) $n > m$, then

$$P(X \geq n | X \geq m) = \frac{P(X \geq n \cap X \geq m)}{P(X \geq m)} = \frac{P(X \geq n)}{P(X \geq m)} = \frac{(1-p)^{n-1}}{(1-p)^{m-1}} = (1-p)^{n-m}.$$

4. You should be able to observe the similarity between this problem and the hashing function: the "candies" are equivalent to the "slots" and the "kids" "keys".

   (a) The probability that a child does not choose a specific candy is $(d-1)/d$, and since each child buys the candies independently, we have $(\frac{d-1}{d})^c$.

   (b) By linearity of expectation, we have $d(\frac{d-1}{d})^c$.

   (c) Remember that when d is large, $(\frac{d-1}{d})^d \to e^{-1}$. Thus, it implies that $1/e$ fraction of all types of candy are not sold to any kid.

5. This question is essentially a reincarnation of coupon collecting problem. The answer is $d \sum_{i=1}^{d} 1/i$.

6. This question is very similar to question 4. Given a specific slot, the probably of its being left empty is $(1 - 1/k)^{2k}$. Again, by linearity of expectation, we have $k(1 - 1/k)^{2k}$ empty slots. When k is large, we know that $(1 - 1/k)^{2k} \to e^{-2}$. So, e^{-2} fraction of the slots will be empty.

7. We know that $H_n = \Theta(\log n)$. Therefore,

$$H_n + H_{n+1} + \cdots + H_2 = \Theta(\log n + \log(n-1) + \cdots \log 2) = O(n \log n).$$

On the other hand, observe that the first half of the terms in the Θ notation is greater than $\log(n/2)$. Therefore, the sum must be greater than some constant times $(n/2)\log(n/2)$, which is $\Omega(n \log n)$. Combining the Ω and the big O, we can conclude the bound is tight.

8. The probability of picking up a value in the middle half is obviously $1/2$. But when there are three "draws", things become more complicated. Let us list the possible scenarios in which this strategy really works. Remember what we want is that the "median" of the three numbers falls in the middle part.

   - All three numbers fall in the middle half. The probability of this is $(1/2)^3$.
   - Two of them fall in the middle half while the remaining falls outside. The probability of this is $\binom{3}{2}(1/2)^2(1/2)$.
   - One of them falls in the middle half. For the remaining two falling outside of the middle half, one must be smaller (thus on the "left" of the middle one) and the other must be greater (on the "right"). The probability of this is $\binom{3}{1}(1/2)\binom{2}{1}(1/4)(1/4)$.

   Summing up all the three above probability, we get $11/16$, which is better than one-half. Thus, in terms of probability, it is really an improvement. However, it is not clear that the potential time savings is worth the extra complexity.

9. We first prove Theorem 5.23.

$$\begin{aligned}
\sum_{i=1}^{n} E(X|F_i)P(F_i) &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_j P(X = x_j|F_i)P(F_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} x_j P(X = x_j \cap F_i) \\
&= \sum_{j=1}^{m} x_j \sum_{i=1}^{n} P(X = x_j \cap F_i) \\
&= \sum_{j=1}^{m} x_j P(X = x_j) \\
&= E(X).
\end{aligned}$$

   For Bayes' theorem, first notice the fact that $A_1, \ldots A_n$ are mutually exclusive and their union composes S. Thus, for the event $B \subseteq S$, it can be decomposed into $P(B) = P(B \cap S) = P(B \cap (A_1 \cup A_2 \cup \cdots A_n)) = P((B \cap A_1) \cup (B \cap A_2) \cup \cdots (B \cap A_n)) = \sum_{k=1}^{n} P(A_k \cap B)$.

   Now we can prove Bayes' Theorem:

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(\bigcup_{k=1}^{n} B \cap A_k)} = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(A_k \cap B)} = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n} P(B|A_k)P(A_k)}.$$