# CS 19: Discrete Mathematics

Prof. Amit Chakrabarti

Today: Relations and Functions

## Relations

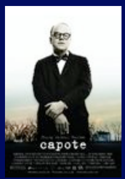
To make sense of the world, we associate or connect things with other things all the time.

- People with people, based on friendship
- People with objects, based on ownership
- Countries with their capitals
- Movies with actors

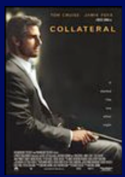
A relation is a mathematical way to connect one set with another set.

(The two sets may in fact be the same.)

## Movies



Capote



Collateral

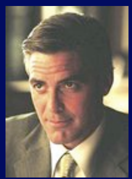

In Her Shoes

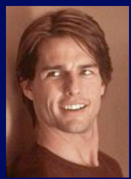

Jarhead

Ray

Vanilla Sky

## Actors


George Clooney


Tom Cruise

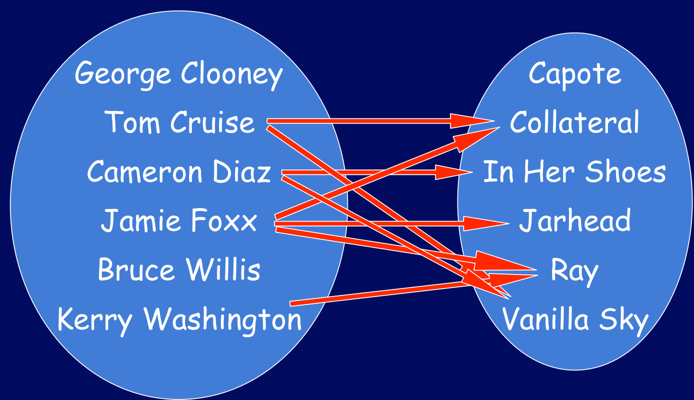

Cameron Diaz


Jamie Foxx

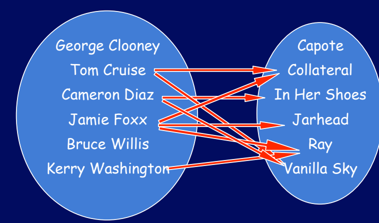

Bruce Willis


Kerry Washington

## The "StarredIn" Relation



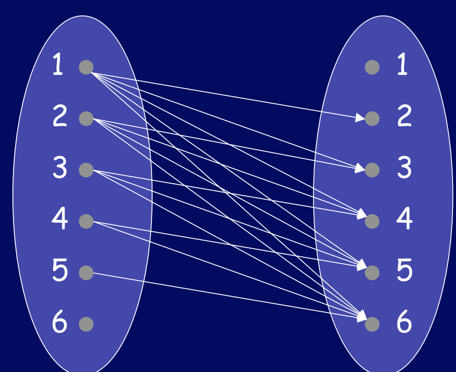

## The "StarredIn" Relation




Describe the relation by listing all pairs of related objects.

(Cruise, Collateral), (Cruise, Vanilla Sky), (Diaz, In Her Shoes), (Diaz, Vanilla Sky), (Foxx, Collateral), (Foxx, Jarhead), (Foxx, Ray), (Washington, Ray)

## The "LessThan" Relation




(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)

## Recall: Cartesian Product

A more sophisticated operation on sets: produces a set of "ordered pairs" (a,b).

$$A \times B = \{ (a,b) : a \in A \text{ and } b \in B \}.$$

| | | | |
|---|---|---|---|
| black | (table, black) | (pen, black) | (hair, black) |
| red | (table, red) | (pen, red) | (hair, red) |
| | table | pen | hair |

e.g., if A = {table, pen, hair}, B = {red, black}, then
A × B = { (table, red), (table, black), (pen, red), (pen, black), (hair, black), (hair, black) }.

## Mathematical Definition of Relation

- A relation from a set A to a set B is a set of ordered pairs of the form (a,b), where $a \in A$ and $b \in B$.
- In other words, it is a subset of $A \times B$.
  - StarredIn $\subseteq$ ACTORS $\times$ MOVIES
  - LessThan $\subseteq \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
- The two sets might in fact be equal: a relation $\mathscr{R} \subseteq A \times A$ is called a relation on A.

## Infix Notation

Let $\mathscr{R} \subseteq A \times B$ be a relation. Instead of writing "$(a,b) \in \mathscr{R}$", we sometimes write "$a \mathscr{R} b$".

E.g., instead of writing "$(2,6) \in$ LessThan", we write "2 LessThan 6", or better still, "$2 < 6$".

Many mathematical symbols that you know express relations. The "<" symbol is an example; as is the "$\in$" symbol.

## The "$\in$" Relation and Power Sets

Let S = {1,2,3}, T = the set of all subsets of S.

i.e., T = { ∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }

We can think of "$\in$" as a relation from S to T given by the following pairs:

(1, {1}), (1, {1,2}), (1, {1,3}), (1, {1,2,3})
(2, {2}), (2, {1,2}), (2, {2,3}), (2, {1,2,3})
(3, {3}), (3, {1,3}), (3, {2,3}), (3, {1,2,3})

BTW, the set T has a special name: it is called the power set of S, denoted P(S).

## Special Types of Relations

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be reflexive if every element $a \in S$ has the property $(a,a) \in R$.

Examples:

- The "knows" relation, on the set of all humans
- The "≤" relation, on the set of all integers (Z)
- The "$\subseteq$" relation, on any power set

## Special Types of Relations

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be reflexive if every element $a \in S$ has the property $(a,a) \in R$.

Non-examples:

- The “bossOf” relation, on the set of humans
- The “$<$” relation, on Z

## Special Types of Relations, II

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be symmetric if every pair of elements $a,b \in S$ has the following property:

if $(a,b) \in R$ then $(b,a) \in R$.

Examples:

- The “knows” relation, on the set of all humans
- The “=” relation, on Z
- The EvenSum relation, on Z, defined as follows: $(x,y) \in$ EvenSum iff $x + y$ is an even number.

## Special Types of Relations, II

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be symmetric if every pair of elements $a,b \in S$ has the following property:

if $(a,b) \in R$ then $(b,a) \in R$.

Non-examples:

- The “knowsOf” relation, on the set of humans
- The “$\leq$” relation, on Z

## Special Types of Relations, III

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be antisymmetric if every pair of elements $a,b \in S$ has the following property:

if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

Examples:

- The “$\leq$” relation, on Z
- The “$\subseteq$” relation, on any power set
- The “$<$” relation, on Z [Why is this??]

## Special Types of Relations, III

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be antisymmetric if every pair of elements $a,b \in S$ has the following property:

if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

Non-examples:

- The "knows" relation, on the set of humans
- The "knowsOf" relation, on the set of humans

## Special Types of Relations, IV

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be transitive if every triplet of elements $a,b,c \in S$ has the following property:

if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

Examples:

- The "sisterOf" relation, on the set of humans
- The "≤" relation, on Z
- The "multipleOf" relation, on Z

## Special Types of Relations, IV

Let S be a set. A relation R on S (i.e., $R \subseteq S \times S$) is said to be transitive if every triplet of elements $a,b,c \in S$ has the following property:

if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

Non-examples:

- The "knows" relation, on the set of humans
- The "≠" relation, on Z

## Functions

Intuitive notion: a function takes an "input" and produces an "output".

Therefore, a function connects a set of "inputs" with a set of "outputs".

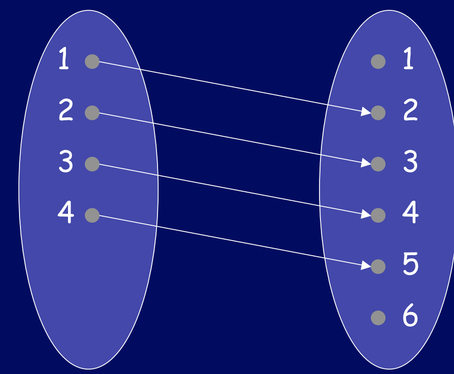
Since it connects two sets, a function is a kind of relation. A very special kind.

## Functions: Definition

A function from a set A to a set B is a relation $f \subseteq A \times B$ with the following property:

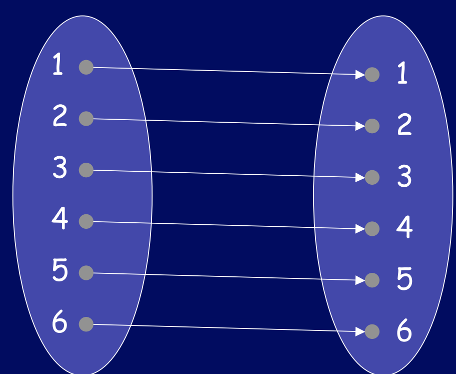
- For every $a \in A$, there exists one and only one $b \in B$ such that $(a,b) \in f$.
- I.e., for every input there is an output and there is only one output.
- It is conventional to write $f(a) = b$, instead of $(a,b) \in f$. Note that $f(a)$ is always uniquely defined.

## Functions: Pictorial



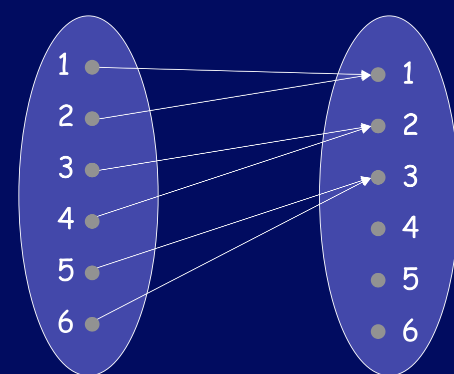

The function $f(a) = a + 1$ from $\{1,2,3,4\}$ to $\{1,2,3,4,5,6\}$

## Functions: Pictorial



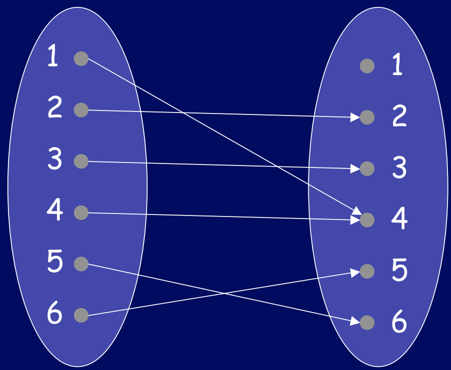

The function $f(a) = a$ from $\{1,2,3,4,5,6\}$ to itself

## Functions: Pictorial



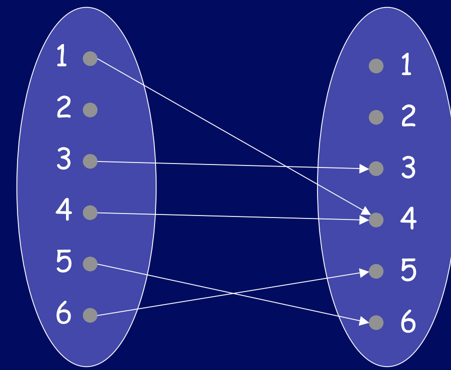

The function $f(a) = \lceil a/2 \rceil$ from $\{1,2,3,4,5,6\}$ to itself

## Functions: Pictorial




A function from {1,2,3,4,5,6} to itself that has no obvious algebraic formula

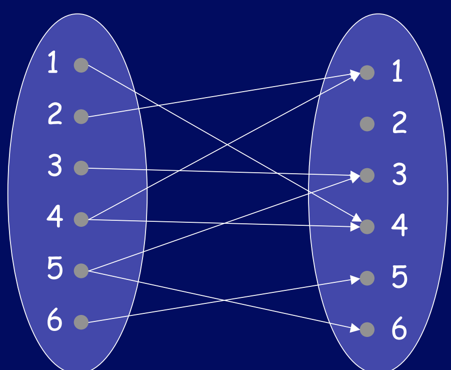
## Not a Function!




Output undefined for input 2.

Mathematically, there is no pair (2,x) in this relation.

## Not a Function!




More than one output for input 4.

There should be only one pair of the form (4,x).

## Functions: More Definitions

Suppose f is a function from A to B. The notation to express this is

$$f : A \to B.$$

- The set A is called the domain of f.
- The set B is called the co-domain of f.
- The subset of B given by $\{f(x) : x \in A\}$ is called the range of f.
  - Note that the range is the set of all actual outputs of f, whereas B is only a set of potential outputs.

## Surjective Functions

Suppose we have a function f : A → B. If every element of B occurs as an actual output of f,

i.e., for every b ∈ B, there exists an a ∈ A such that f(a) = b,

i.e., there is an arrow leading in to every element of B,

then we say that

- f is surjective.
- f is a surjection.
- f is onto.

## Injective Functions

Suppose we have a function f : A → B. If no two elements of A produce the same output,

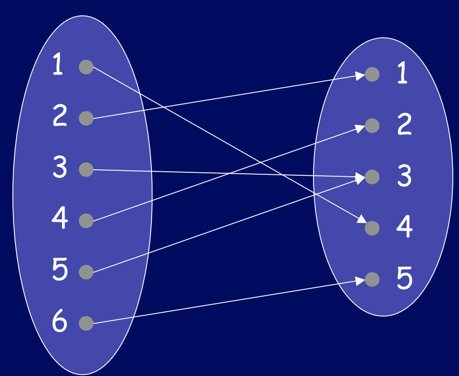
i.e., for every x,y ∈ A, if x ≠ y then f(x) ≠ f(y),

i.e., arrows from different elements of A lead in to different elements of B,

then we say that

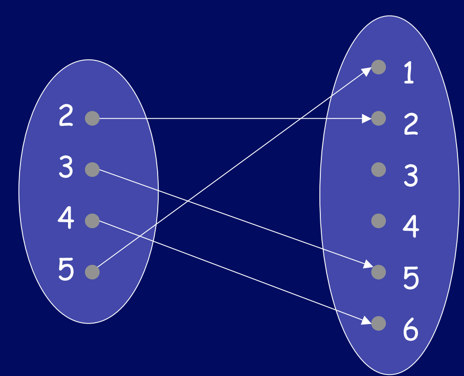
- f is injective.
- f is an injection.
- f is one-to-one.

## A Surjective Function




But not injective, because f(3) = f(5).

## An Injective Function




But not surjective, because there is no x such that f(x) = 3

## More Examples

Let Z be the set of all integers

- Is the function f : Z → Z given by f(x) = 2x injective? Is it surjective?
  - Injective, but not surjective
- How about f : Z → Z given by f(x) = x + 1 ?
  - Both injective and surjective
- How about f : Z → Z given by f(x) = ⌊x/5⌋ ?
  - Surjective, but not injective
- How about f : Z → Z given by f(x) = |x|?
  - Neither injective nor surjective

Study Bee

Concepts:

- Relations
- Reflexive, symmetric, antisymmetric and transitive
- Functions
- Injective and surjective

Examples:

- Lots of examples of all the concepts we've defined today.
- Study them carefully!
- Think up more examples.