Additional Problems for Homework
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***** Do not forget to hit RELOAD *****

AP1 Figure out the sum of the *odd* binomial coefficients. To be precise, work out the sum

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \binom{n}{i} .$$

If you are stuck for ideas, look at Problem 17 in Section 1.3 of the textbook.

AP2 Suppose n lines are drawn in the plane with no two of them parallel and no three of them concurrent. We showed in class that such an arrangement of lines partitions the plane into $n(n+1)/2+1$ regions. Prove by induction that it is possible to color these regions using two colors, say black and white, such that no two adjacent regions are colored the same.

[*As an interesting aside, there is a famous theorem which shows that any "map" drawn on the plane can be colored using at most four colors such that no two adjacent "countries" are colored the same. This is the* ***Four Color Theorem****, famous for having stumped mathematicians for over 120 years, and for having one of the most notoriously complicated proofs in discrete mathematics. You can think of this problem as a very, very simple first step towards that great theorem.*]

AP3 Poker is a rather mathematical card game with many variations, but all based on the basic principle that certain hands "beat" (i.e., have higher value than) certain others. A hand consists of five distinct cards chosen from a standard deck of 52 cards. There are nine special hands and they are given specific names, as follows.

A *royal flush* consists of the Ace, King, Queen, Jack and 10, all of the same suit. A *straight flush* is any five-card sequence within a suit, except for the one beginning with the Ace (that would make it royal), for instance, Jack, 10, 9, 8, 7. A *straight* is any five-card sequence with not all cards of the same suit and a *flush* is a set of five cards of the same suit but not in sequence.

A *four-of-a-kind* is hand with four cards of the same value (e.g. with four 9s). A *three-of-a-kind* has three cards of the same value and two other cards with two other different values; had these two other cards had the same value that would give us a *full house*; thus, three 5s, a King and a 9 give us a three-of-a-kind whereas three 5s and two 10s give us a full house. A *two pair* contains two different equal-value pairs and an unrelated fifth card, e.g. two 7s, two Kings, and an Ace. Finally, a *pair* has just one equal-value pair and three other unrelated cards.

(a) Suppose you shuffle a deck of 52 cards and then draw five cards at random. What is the probability of getting each of the nine special hands (royal flush, straight flush, straight, flush, four-of-a-kind, three-of-a-kind, full house, two pair and pair)?

(b) A sensible design of the rules of poker would ensure that if you have been lucky enough to draw an "unlikely" hand, then your hand should beat that of an opponent who has drawn a more "mundane" hand. In fact, poker *was* sensibly designed. By arranging the hands from least likely to most likely, figure out the "pecking order" of these special hands in poker. You might want to use a calculator unless you're a whiz with numbers!

AP4 Find the number of integers between 1 and $10{,}000$ inclusive, which are not divisible by 4, 5, or 6.

AP5 This problem should convince you not to trust vaguely formed "intuitions" about probability, but instead to carefully work out the numbers using the proper definitions and theorems from probability theory. It is the famous (some would say infamous) Monty Hall problem, which gets its name from the TV game show *Let's Make A Deal*, hosted by Monty Hall.

You are asked to select one closed door of three, behind one of which there is a prize. The other two doors hide nothing. Once you have made your selection, Monty Hall opens one of the remaining doors, revealing that it does not contain the prize. He then asks you if you would like to switch your selection to the other unopened door, or stay with your original choice. The problem: should you switch?

Work this out meticulously. Carefully define a sample space, define any necessary events and then work out two probabilities:

(a) The probability that you win the prize if you don't switch.

(b) The probability that you win the prize if you do switch.

Do you find the answers intuitive? (There is no incorrect answer to that question!) If not, the lesson you have learnt is that you need to wait until your intuition has matured before trusting it.

AP6 Consider the recurrence

$$T(n) \;=\; \begin{cases} 4T(n/2) + n\log n, & \text{if } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

Prove that $T(n) = \Theta(n^2)$, for n a power of 2. You can use any method you like, such as the recursion tree method from the textbook, or the substitution method (i.e., let $n = 2^k$) from class. Note that the master theorem does not apply, but you should be able to mimic its proof. At some stage you may find it helpful to look at Problem 3 from the First Midterm.

AP7 Let α, β and c be positive real constants with $\alpha + \beta < 1$. Suppose $T(n)$ is a sequence defined on the integers that satisfies the inequality

$$T(n) \;\le\; T(\lceil \alpha n \rceil) + T(\lceil \beta n \rceil) + cn\,.$$

Give a careful proof, using induction and the precise definition of big-O, that $T(n) = O(n)$.

AP8 Using your favorite programming language, implement the extended GCD algorithm and the inverse algorithm. Then use this program to try to find

(a) the inverse of 80966 modulo 3469391, and

(b) the inverse of 277650 modulo 3469391.

Chances are you've already written a similar program for CS 18. You do not need to submit your source code (but do so if you really like your program); just the answers will do.

AP9 *A formula for $\phi(n)$:* In this problem we shall derive a neat formula for $\phi(n)$, based on the prime factorization of n. Let U_n denote the set of numbers in $\{1, 2, \ldots, n-1\}$ that are relatively prime to n. By definition, $\phi(n) = |U_n|$.

1. Suppose m and n are relatively prime. Show that, for any $i \in U_{mn}$, the integer $i \bmod m$ is relatively prime to m and the integer $i \bmod n$ is relatively prime to n. Conclude that $i \bmod m \in U_m$ and $i \bmod n \in U_n$.
2. For the above m and n, define a function $\pi : U_{mn} \to U_m \times U_n$ by

   $$\pi(i) \;=\; (i \bmod m, i \bmod n)\,,$$

   for each $i \in U_{mn}$. Using the Chinese Remainder Theorem (or mimicking its proof), prove that π is a bijection. Here $U_m \times U_n$ is the Cartesian product, as defined in Problem 1 of Midterm 1.

3. Conclude that, for any two relatively prime integers m and n, we have $\phi(mn) = \phi(m)\phi(n)$.
4. Using mathematical induction, prove that if the numbers $m_1, m_2, \ldots, m_k$ are *pairwise* relatively prime, then $\phi(m_1 m_2 \cdots m_k) = \phi(m_1)\phi(m_2)\cdots\phi(m_k)$.
5. It is easy to see that if p is a prime, $\phi(p) = p - 1$. Show that $\phi(p^2) = p^2 - p$. Generalize your argument to prove that $\phi(p^a) = p^a - p^{a-1} = p^a(1 - 1/p)$, for any positive integer a.
6. Now suppose the prime factorization of n is given by

$$n \;=\; p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \,,$$

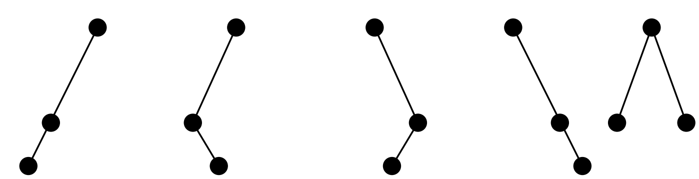
where the p_i are distinct primes and the a_i are positive integers. For instance, if you were to write the prime factorization of 1200 in this form, you would write $1200 = 2^4 \times 3 \times 5^2$, so you would have $k = 3$, $(p_1, p_2, p_3) = (2, 3, 5)$, and $(a_1, a_2, a_3) = (4, 1, 2)$. Put together all of the pieces you proved above to conclude that

$$\phi(n) \;=\; n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) .$$

7. Now that you have this brand new toy, use it to compute $\phi(1200)$ and $\phi(16335)$. You will probably want to use a calculator.

A bonus question: Now that you know a formula for $\phi(n)$, can you stare at it and think of another way we could have derived it? Think of what happens when you open the parentheses. You don't have to turn in an answer to this question, but if you get it, let the instructor know.

AP10 The number of rooted binary trees with n vertices, denoted C_n, is a quantity that pops up in many counting problems. These numbers come up in so many different situations that they're given a special name. They're called the *Catalan numbers*. Clearly $C_1 = 1$. By convention, we define $C_0 = 1$ as well. Verify for yourself that $C_2 = 2$. The following figure shows that $C_3 = 5$.



The five binary trees with that have three vertices each

Prove that the Catalan numbers satisfy the following recurrence.

$$C_n \;=\; \begin{cases} \sum_{k=1}^{n} C_{k-1}C_{n-k}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

Using this recurrence, compute C_8 and C_9.

It is possible to solve this recurrence and obtain

$$C_n \;=\; \frac{1}{n+1}\binom{2n}{n} .$$

Verify this formula for $n = 8$ and $n = 9$.

I know of three very different ways to solve the recurrence for C_n, each of them very beautiful. Unfortunately, none of the three methods is straightforward and each of them would take us too far afield from graph theory.

AP11 A graph is said to be k-pseudoregular if every vertex of the graph has degree either k or $k+1$. Prove that, for $k \geq 1$, every k-pseudoregular graph has a $(k-1)$-pseudoregular subgraph. Hint: Find a suitable matching in the graph and delete the edges in this matching.