Preface

This book grew out of lecture notes for offerings of a course on data stream algorithms at Dartmouth, beginning with a first offering in Fall 2009. Its primary goal is to be a resource for students and other teachers of this material. The choice of topics reflects this: the focus is on foundational algorithms for space-efficient processing of massive data streams. I have emphasized algorithms that are simple enough to permit a clean and fairly complete presentation in a classroom, assuming not much more background than an advanced undergraduate student would have. Where appropriate, I have provided pointers to more advanced research results that achieve improved bounds using heavier technical machinery.

I would like to thank the many Dartmouth students who took various editions of my course, as well as researchers around the world who told me that they had found my course notes useful. Their feedback inspired me to make a book out of this material. I thank Suman Kalyan Bera, Sagar Kale, and Jelani Nelson for numerous comments and contributions. Of special note are the 2009 students whose scribing efforts got this book started: Radhika Bhasin, Andrew Cherne, Robin Chhetri, Joe Cooley, Jon Denning, Alina Djamankulova, Ryan Kingston, Ranganath Kondapally, Adrian Kostrubiak, Konstantin Kutzkow, Aarathi Prasad, Priya Natarajan, and Zhenghui Wang.

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Preliminaries: The Data Stream Model

0.1 The Basic Setup

In this course, we are concerned with algorithms that compute some function of a massively long input stream $\sigma$. In the most basic model (which we shall call the vanilla streaming model), this is formalized as a sequence $\sigma = \langle a_1, a_2, \ldots, a_m \rangle$, where the elements of the sequence (called tokens) are drawn from the universe $[n] := \{1, 2, \ldots, n\}$. Note the two important size parameters: the stream length, $m$, and the universe size, $n$. If you read the literature in the area, you will notice that some authors interchange these two symbols. In this course, we shall consistently use $m$ and $n$ as we have just defined them.

Our central goal will be to process the input stream using a small amount of space $s$, i.e., to use $s$ bits of random-access working memory. Since $m$ and $n$ are to be thought of as “huge,” we want to make $s$ much smaller than these; specifically, we want $s$ to be sublinear in both $m$ and $n$. In symbols, we want

$$s = o\left(\min\{m,n\}\right).$$ (1)

The holy grail is to achieve

$$s = O(\log m + \log n),$$ (2)

because this amount of space is what we need to store a constant number of tokens from the stream and a constant number of counters that can count up to the length of the stream. Sometimes we can only come close and achieve a space bound of the form $s = \text{polylog}(\min\{m,n\})$, where $f(n) = \text{polylog}(g(n))$ means that there exists a constant $c > 0$ such that $f(n) = O((\log g(n))^c)$.

The reason for calling the input a stream is that we are only allowed to access the input in “streaming fashion.” That is, we do not have random access to the tokens and we can only scan the sequence in the given order. We do consider algorithms that make $p$ passes over the stream, for some “small” integer $p$, keeping in mind that the holy grail is to achieve $p = 1$. As we shall see, in our first few algorithms, we will be able to do quite a bit in just one pass.

0.2 The Quality of an Algorithm’s Answer

The function we wish to compute—$\phi(\sigma)$, say—is often real-valued. We shall typically seek to compute only an estimate or approximation of the true value of $\phi(\sigma)$, because many basic functions can provably not be computed exactly using sublinear space. For the same reason, we shall often allow randomized algorithms than may err with some small, but controllable, probability. This motivates the following basic definition.

**Definition 0.2.1.** Let $\mathcal{A}(\sigma)$ denote the output of a randomized streaming algorithm $\mathcal{A}$ on input $\sigma$; note that this is a random variable. Let $\phi$ be the function that $\mathcal{A}$ is supposed to compute. We say that the algorithm $(\epsilon, \delta)$-estimates $\phi$ if
we have
\[ \Pr \left[ \left| \frac{\phi'(\sigma)}{\phi(\sigma)} - 1 \right| > \epsilon \right] \leq \delta. \]

Notice that the above definition insists on a multiplicative approximation. This is sometimes too strong a condition when the value of \( \phi(\sigma) \) can be close to, or equal to, zero. Therefore, for some problems, we might instead seek an additive approximation, as defined below.

**Definition 0.2.2.** In the above setup, the algorithm \( \mathcal{A} \) is said to \( (\epsilon, \delta) \)-estimate \( \phi \) if we have
\[ \Pr \left[ \left| \mathcal{A}(\sigma) - \phi(\sigma) \right| > \epsilon \right] \leq \delta. \]

We have mentioned that certain things are provably impossible in sublinear space. Later in the course, we shall study how to prove such impossibility results. Such impossibility results, also called lower bounds, are a rich field of study in their own right.

### 0.3 Variations of the Basic Setup

Quite often, the function we are interested in computing is some statistical property of the multiset of items in the input stream \( \sigma \). This multiset can be represented by a frequency vector \( \mathbf{f} = (f_1, f_2, \ldots, f_n) \), where
\[ f_j = |\{i: a_i = j\}| = \text{number of occurrences of } j \text{ in } \sigma. \]

In other words, \( \sigma \) implicitly defines this vector \( \mathbf{f} \), and we are then interested in computing some function of the form \( \Phi(\mathbf{f}) \). While processing the stream, when we scan a token \( j \in [n] \), the effect is to increment the frequency \( f_j \). Thus, \( \sigma \) can be thought of as a sequence of update instructions, updating the vector \( \mathbf{f} \).

With this in mind, it is interesting to consider more general updates to \( \mathbf{f} \): for instance, what if items could both “arrive” and “depart” from our multiset, i.e., if the frequencies \( f_j \) could be both incremented and decremented, and by variable amounts? This leads us to the turnstile model, in which the tokens in \( \sigma \) belong to \( [n] \times \{-L, \ldots, L\} \), interpreted as follows:

Upon receiving token \( a_i = (j, c) \), update \( f_j \leftarrow f_j + c \).

Naturally, the vector \( \mathbf{f} \) is assumed to start out at \( \mathbf{0} \). In this generalized model, it is natural to change the role of the parameter \( m \): instead of the stream’s length, it will denote the maximum number of items in the multiset at any point of time. More formally, we require that, at all times, we have
\[ ||\mathbf{f}||_1 = |f_1| + \cdots + |f_n| \leq m. \]

A special case of the turnstile model, that is sometimes important to consider, is the strict turnstile model, in which we assume that \( \mathbf{f} \geq 0 \) at all times. A further special case is the cash register model, where we only allow positive updates: i.e., we require that every update \( (j, c) \) have \( c > 0 \).
Finding Frequent Items Deterministically

Our study of data stream algorithms begins with statistical problems: the input stream describes a multiset of items and we would like to be able to estimate certain statistics of the empirical frequency distribution (see Section 0.3). We shall study a number of such problems over the next several units. In this unit, we consider one such problem that admits a particularly simple and deterministic (yet subtle and powerful) algorithm.

1.1 The Problem

We are in the vanilla streaming model. We have a stream \( \sigma = \langle a_1, \ldots, a_n \rangle \), with each \( a_i \in [n] \), and this implicitly defines a frequency vector \( f = (f_1, \ldots, f_n) \). Note that \( f_1 + \cdots + f_n = m \).

In the MAJORITY problem, our task is as follows. If \( \exists j : f_j > m/2 \), then output \( j \), otherwise, output “⊥”.

This can be generalized to the FREQUENT problem, with parameter \( k \), as follows: output the set \( \{ j : f_j > m/k \} \).

In this unit, we shall limit ourselves to deterministic algorithms for this problem. If we further limit ourselves to one-pass algorithms, even the simpler problem, MAJORITY, provably requires \( \Omega(\min\{m, n\}) \) space. However, we shall soon give a one-pass algorithm—the Misra–Gries Algorithm [MG82]—that solves the related problem of estimating the frequencies \( f_j \). As we shall see,

1. the properties of Misra–Gries are interesting in and of themselves, and
2. it is easy to extend Misra–Gries, using a second pass, to then solve the FREQUENT problem.

Thus, we now turn to the FREQUENCY-ESTIMATION problem. The task is to process \( \sigma \) to produce a data structure that can provide an estimate \( \hat{f}_a \) for the frequency \( f_a \) of a given token \( a \in [n] \). Note that the query \( a \) is given to us only after we have processed \( \sigma \). This problem can also be thought of as querying the frequency vector \( f \) at the point \( a \), so it is also known as the POINT-QUERY problem.

1.2 Frequency Estimation: The Misra–Gries Algorithm

As with all one-pass data stream algorithms, we shall have an initialization section, executed before we see the stream, a processing section, executed each time we see a token, and an output section, where we answer question(s) about the stream, perhaps in response to a given query.

This algorithm (see Algorithm 1) uses a parameter \( k \) that controls the quality of the answers it gives. (Looking ahead: to solve the FREQUENT problem with parameter \( k \), we shall run the Misra–Gries algorithm with parameter \( k \).) It maintains an associative array, \( A \), whose keys are tokens seen in the stream, and whose values are counters associated with these tokens. We keep at most \( k - 1 \) counters at any time.
Algorithm 1 The Misra–Gries frequency estimation algorithm

Initialize:
1: \( A \leftarrow \) (empty associative array)

Process (token \( j \)):
2: if \( j \in \text{keys}(A) \) then
3: \( A[j] \leftarrow A[j] + 1 \)
4: else if \( |\text{keys}(A)| < k - 1 \) then
5: \( A[j] \leftarrow 1 \)
6: else
7: foreach \( \ell \in \text{keys}(A) \) do
8: \( A[\ell] \leftarrow A[\ell] - 1 \)
9: if \( A[\ell] = 0 \) then remove \( \ell \) from \( A \)

Output (query \( a \)):
10: if \( a \in \text{keys}(A) \) then report \( \hat{f}_a = A[a] \) else report \( \hat{f}_a = 0 \)

1.3 Analysis of the Algorithm

To process each token quickly, we could maintain the associative array \( A \) using a balanced binary search tree. Each key requires \( \lceil \log n \rceil \) bits to store and each value requires at most \( \lceil \log m \rceil \) bits. Since there are at most \( k - 1 \) key/value pairs in \( A \) at any time, the total space required is \( O(k(\log m + \log n)) \).

We now analyze the quality of the algorithm’s output.

Pretend that \( A \) consists of \( n \) key/value pairs, with \( A[j] = 0 \) whenever \( j \) is not actually stored in \( A \) by the algorithm. Consider the increments and decrements to \( A[j] \)'s as the stream comes in. For bookkeeping, pretend that upon entering the loop at line 7, \( A[j] \) is incremented from 0 to 1, and then immediately decremented back to 0. Further, noting that each counter \( A[j] \) corresponds to a set of occurrences of \( j \) in the input stream, consider a variant of the algorithm (see Algorithm 2) that explicitly maintains this set. Of course, actually doing so is horribly space-inefficient!

Algorithm 2 Thought-experiment algorithm for analysis of Misra–Gries

Initialize:
1: \( B \leftarrow \) (empty associative array)

Process (stream-position \( i \), token \( j \)):
2: if \( j \in \text{keys}(B) \) then
3: \( B[j] \leftarrow B[j] \cup \{i\} \)
4: else
5: \( B[j] \leftarrow \{i\} \)
6: if \( |\text{keys}(B)| = k \) then
7: foreach \( \ell \in \text{keys}(B) \) do
8: \( B[\ell] \leftarrow B[\ell] \setminus \text{min}(B[\ell]) \) \( \triangleright \) forget earliest occurrence of \( \ell \)
9: if \( B[\ell] = \emptyset \) then remove \( \ell \) from \( B \)

Output (query \( a \)):
10: if \( a \in \text{keys}(B) \) then report \( \hat{f}_a = |B[a]| \) else report \( \hat{f}_a = 0 \)

Notice that after each token is processed, each \( A[j] \) is precisely the cardinality of the corresponding set \( B[j] \). The increments to \( A[j] \) (including the aforementioned pretend ones) correspond precisely to the occurrences of \( j \) in the stream. Thus, \( \hat{f}_j \leq f_j \).

On the other hand, decrements to counters (including the pretend ones) occur in sets of \( k \). To be precise, referring to Algorithm 2, whenever the sets \( B[\ell] \) are shrunk, exactly \( k \) of the sets shrink by one element each and the removed
elements are \( k \) distinct stream positions. Focusing on a particular item \( j \), each decrement of \( A[j] \) is “witnessed” by a collection of \( k \) distinct stream positions. Since the stream length is \( m \), there can be at most \( m/k \) such decrements. Therefore, \( \hat{f}_j \geq f_j - m/k \). Putting these together we have the following theorem.

**Theorem 1.3.1.** The Misra–Gries algorithm with parameter \( k \) uses one pass and \( O(k(\log m + \log n)) \) bits of space, and provides, for any token \( j \), an estimate \( \hat{f}_j \) satisfying

\[
f_j - \frac{m}{k} \leq \hat{f}_j \leq f_j.
\]

## 1.4 Finding the Frequent Items

Using the Misra–Gries algorithm, we can now easily solve the FREQUENT problem in one additional pass. By the above theorem, if some token \( j \) has \( f_j > m/k \), then its corresponding counter \( A[j] \) will be positive at the end of the Misra–Gries pass over the stream, i.e., \( j \) will be in \( \text{keys}(A) \). Thus, we can make a second pass over the input stream, counting exactly the frequencies \( f_j \) for all \( j \in \text{keys}(A) \), and then output the desired set of items.

Alternatively, if limited to a single pass, we can solve FREQUENT in an approximate sense: we may end up outputting items that are below (but not too far below) the frequency threshold. We explore this in the exercises.

### Exercises

1-1 Let \( \tilde{m} \) be the sum of all counters maintained by the Misra–Gries algorithm after it has processed an input stream, i.e., \( \tilde{m} = \sum_{\ell \in \text{keys}(A)} A[\ell] \). Prove that the bound in Theorem 1.3.1 can be sharpened to

\[
f_j - \frac{m - \tilde{m}}{k} \leq \hat{f}_j \leq f_j. \tag{1.1}
\]

1-2 Items that occur with high frequency in a dataset are sometimes called heavy hitters. Accordingly, let us defined the HEAVY-HITTERS problem, with real parameter \( \varepsilon > 0 \), as follows. The input is a stream \( \sigma \). Let \( m, n, f \) have their usual meanings. Let

\[
\text{HH}_\varepsilon(\sigma) = \{ j \in [n] : f_j \geq \varepsilon m \}
\]

be the set of \( \varepsilon \)-heavy hitters in \( \sigma \). Modify Misra–Gries to obtain a one-pass streaming algorithm that outputs this set “approximately” in the following sense: the set \( H \) it outputs should satisfy

\[
\text{HH}_\varepsilon(\sigma) \subseteq H \subseteq \text{HH}_{\varepsilon/2}(\sigma).
\]

Your algorithm should use \( O(\varepsilon^{-1}(\log m + \log n)) \) bits of space.

1-3 Suppose we have run the (one-pass) Misra–Gries algorithm on two streams \( \sigma_1 \) and \( \sigma_2 \), thereby obtaining a summary for each stream consisting of \( k \) counters. Consider the following algorithm for merging these two summaries to produce a single \( k \)-counter summary.

1: Combine the two sets of counters, adding up counts for any common items.
2: If more than \( k \) counters remain:
   1. \( c \leftarrow \) value of \((k+1)\)th counter, based on decreasing order of value.
   2. Reduce each counter by \( c \) and delete all keys with non-positive counters.

Prove that the resulting summary is good for the combined stream \( \sigma_1 \circ \sigma_2 \) (here “\( \circ \)” denotes concatenation of streams) in the sense that frequency estimates obtained from it satisfy the bounds given in Eq. (1.1).
Estimating the Number of Distinct Elements

We continue with our study of data streaming algorithms for statistical problems. Previously, we focused on identifying items that are particularly dominant in a stream, appearing with especially high frequency. Intuitively, we could solve this in sublinear space because only a few items can be dominant and we can afford to throw away information about non-dominant items. Now, we consider a very different statistic: namely, **how many** distinct tokens (elements) appear in the stream. This is a measure of how “spread out” the stream is. It is not intuitively clear that we can estimate this quantity well in sublinear space, because we can’t afford to ignore rare items. In particular, merely sampling some tokens from the stream will mislead us, since a sample will tend to pick up frequent items rather than rare ones.

### 2.1 The Problem

As in Unit 1, we are in the vanilla streaming model. We have a stream \( \sigma = (a_1, \ldots, a_n) \), with each \( a_i \in [n] \), and this implicitly defines a frequency vector \( f = (f_1, \ldots, f_n) \). Let \( d = |\{j : f_j > 0\}| \) be the number of distinct elements that appear in \( \sigma \).

In the **DISTINCT-ELEMENTS** problem, our task is to output an \((\varepsilon, \delta)\)-estimate (as in Definition 0.2.1) to \( d \).

It is provably impossible to solve this problem in sublinear space if one is restricted to either deterministic algorithms (i.e., \( \delta = 0 \)), or exact algorithms (i.e., \( \varepsilon = 0 \)). Thus, we shall seek a randomized approximation algorithm. In this unit, we give a simple algorithm for this problem that has interesting, but not optimal, quality guarantees. Despite being sub-optimal, it is worth studying because

- the algorithm is especially simple;
- it introduces us to two ingredients used in tons of randomized algorithms, namely, *universal hashing* and the *median trick*;
- it introduces us to probability tail bounds, a basic technique for the analysis of randomized algorithms.

### 2.2 The Tidemark Algorithm

The idea behind the algorithm is originally due to Flajolet and Martin [FM85]. We give a slightly modified presentation, due to Alon, Matias and Szegedy [AMS99]. Since that paper designs several other algorithms as well (for other problems), it’s good to give this particular algorithm a name more evocative than “AMS algorithm.” I call it the *tidemark algorithm* because of how it remembers information about the input stream. Metaphorically speaking, each token has an opportunity to raise the “water level” and the algorithm simply keeps track of the high-water mark, just as a tidemark records the high-water mark left by tidal water.
For an integer \( p > 0 \), let \( \text{zeros}(p) \) denote the number of zeros that the binary representation of \( p \) ends with. Formally,
\[
\text{zeros}(p) = \max \{ i : 2^i \text{ divides } p \}.
\]

Our algorithm’s key ingredient is a 2-universal hash family, a very important concept that will come up repeatedly. If you are unfamiliar with the concept, working through Exercises 2-1 and 2-2 is strongly recommended. Once we have this key ingredient, our algorithm is very simple.

**Algorithm 3** The tidemark algorithm for the number of distinct elements

**Initialize:**
1. Choose a random hash function \( h : [n] \to [n] \) from a 2-universal family
2. \( z \leftarrow 0 \)

**Process** \((\text{token } j)\):
3. if \( \text{zeros}(h(j)) > z \) then \( z \leftarrow \text{zeros}(h(j)) \)

**Output:** \( 2^{z+\frac{1}{2}} \)

The basic intuition here is that we expect 1 out of the \( d \) distinct tokens to hit \( \text{zeros}(h(j)) \geq \log d \), and we don’t expect any tokens to hit \( \text{zeros}(h(j)) \gg \log d \). Thus, the maximum value of \( \text{zeros}(h(j)) \) over the stream—which is what we maintain in \( z \)—should give us a good approximation to \( \log d \). We now analyze this.

### 2.3 The Quality of the Algorithm’s Estimate

Formally, for each \( j \in [n] \) and each integer \( r \geq 0 \), let \( X_{r,j} \) be an indicator random variable for the event “\( \text{zeros}(h(j)) \geq r \),” and let \( Y_r = \sum_{j : f_j > 0} X_{r,j} \). Let \( T \) denote the value of \( z \) when the algorithm finishes processing the stream. Clearly,
\[
Y_r > 0 \iff T \geq r.
\]

We can restate the above fact as follows (this will be useful later):
\[
Y_r = 0 \iff T \leq r - 1.
\]

Since \( h(j) \) is uniformly distributed over the \((\log n)\)-bit strings, we have
\[
\mathbb{E} X_{r,j} = \mathbb{P}\{\text{zeros}(h(j)) \geq r\} = \mathbb{P}\{2^r \text{ divides } h(j)\} = \frac{1}{2^r}.
\]

We now estimate the expectation and variance of \( Y_r \) as follows. The first step of Eq. (2.3) below uses the pairwise independence of the random variables \( \{X_{r,j}\}_{j \in [n]} \), which follows from the 2-universality of the hash family from which \( h \) is drawn.
\[
\mathbb{E} Y_r = \sum_{j : f_j > 0} \mathbb{E} X_{r,j} = \frac{d}{2^r}.
\]
\[
\text{Var} Y_r = \sum_{j : f_j > 0} \text{Var} X_{r,j} \leq \sum_{j : f_j > 0} \mathbb{E} (X_{r,j}^2) = \sum_{j : f_j > 0} \mathbb{E} X_{r,j} = \frac{d}{2^r}.
\]

Thus, using Markov’s and Chebyshev’s inequalities respectively, we have
\[
\mathbb{P}\{Y_r > 0\} = \mathbb{P}\{Y_r \geq 1\} \leq \frac{\mathbb{E} Y_r}{1} = \frac{d}{2^r} \quad \text{and} \quad \mathbb{P}\{Y_r = 0\} \leq \mathbb{P}\{|Y_r - \mathbb{E} Y_r| \geq d/2^r\} \leq \frac{\text{Var} Y_r}{(d/2^r)^2} \leq \frac{2^r}{d}.
\]
Let \( \hat{d} \) be the estimate of \( d \) that the algorithm outputs. Then \( \hat{d} = 2^{T + \frac{1}{2}} \). Let \( a \) be the smallest integer such that \( 2^{a + \frac{1}{2}} \geq 3d \). Using eqs. (2.1) and (2.4), we have
\[
    P \{ \hat{d} \geq 3d \} = P \{ T \geq a \} = P \{ Y_a > 0 \} \leq \frac{d}{2^a} \leq \frac{\sqrt{2}}{3}.
\]
(2.6)

Similarly, let \( b \) be the largest integer such that \( 2^{b + \frac{1}{2}} \leq d/3 \). Using Eqs. (2.2) and (2.5), we have
\[
    P \{ \hat{d} \leq d/3 \} = P \{ T \leq b \} = P \{ Y_{b+1} = 0 \} \leq \frac{2^{b + 1}}{d} \leq \frac{\sqrt{2}}{3}.
\]
(2.7)

These guarantees are weak in two ways. Firstly, the estimate \( \hat{d} \) is only of the “same order of magnitude” as \( d \), and is not an arbitrarily good approximation. Secondly, these failure probabilities in eqs. (2.6) and (2.7) are only bounded by the rather large \( \sqrt{2}/3 \approx 47\% \). Of course, we could make the probabilities smaller by replacing the constant “3” above with a larger constant. But a better idea, that does not further degrade the quality of the estimate \( \hat{d} \), is to use a standard “median trick” which will come up again and again.

2.4 The Median Trick

Imagine running \( k \) copies of this algorithm in parallel, using mutually independent random hash functions, and outputting the median of the \( k \) answers. If this median exceeds \( 3d \), then at least \( k/2 \) of the individual answers must exceed \( 3d \), whereas we only expect \( k \sqrt{2}/3 \) of them to exceed \( 3d \). By a standard Chernoff bound, this event has probability \( 2^{-\Omega(k)} \). Similarly, the probability that the median is below \( d/3 \) is also \( 2^{-\Omega(k)} \).

Choosing \( k = \Theta(\log(1/\delta)) \), we can make the sum of these two probabilities work out to at most \( \delta \). This gives us an \( (O(1), \delta) \)-estimate for \( d \). Later, we shall give a different algorithm that will provide an \( (\epsilon, \delta) \)-estimate with \( \epsilon \to 0 \).

The original algorithm requires \( O(\log n) \) bits to store (and compute) a suitable hash function, and \( O(\log \log n) \) more bits to store \( z \). Therefore, the space used by this final algorithm is \( O(\log(1/\delta) \cdot \log n) \). When we reattack this problem with a new algorithm, we will also improve this space bound.

Exercises

These exercises are designed to get you familiar with the very important concept of a 2-universal hash family, as well as give you constructive examples of such families.

Let \( X \) and \( Y \) be finite sets and let \( Y^X \) denote the set of all functions from \( X \) to \( Y \). We will think of these functions as “hash” functions. [The term “hash function” has no formal meaning; strictly speaking, one should say “family of hash functions” or “hash family” as we do here.] A family \( \mathcal{H} \subseteq Y^X \) is said to be 2-universal if the following property holds, with \( h \in_R \mathcal{H} \) picked uniformly at random:
\[
    \forall x, x' \in X \forall y, y' \in Y \left( x \neq x' \Rightarrow P_h \{ h(x) = y \land h(x') = y' \} = \frac{1}{|Y|^2} \right).
\]

We shall give two examples of 2-universal hash families from the set \( X = \{0,1\}^n \) to the set \( Y = \{0,1\}^k \) (with \( k \leq n \)).

2-1 Treat the elements of \( X \) and \( Y \) as column vectors with \( 0/1 \) entries. For a matrix \( A \in \{0,1\}^{k \times n} \) and vector \( b \in \{0,1\}^k \), define the function \( h_{A,b} : X \to Y \) by \( h_{A,b}(x) = Ax + b \), where all additions and multiplications are performed mod 2.

Prove that the family of functions \( \mathcal{H} = \{ h_{A,b} : A \in \{0,1\}^{k \times n}, b \in \{0,1\}^k \} \) is 2-universal.
2-2 Identify $X$ with the finite field $\mathbb{F}_{2^n}$ using an arbitrary bijection—truly arbitrary: e.g., the bijection need not map the string $0^n$ to the zero element of $\mathbb{F}_{2^n}$. For elements $a, b \in X$, define the function $g_{a,b} : X \rightarrow Y$ as follows:

\[
g_{a,b}(x) = \text{rightmost } k \text{ bits of } f_{a,b}(x), \quad \text{where}
\]
\[
f_{a,b}(x) = ax + b, \quad \text{with addition and multiplication performed in } \mathbb{F}_{2^n}.
\]

Prove that the family of functions $\mathcal{G} = \{g_{a,b} : a, b \in \mathbb{F}_{2^n}\}$ is 2-universal. Is the family $\mathcal{G}$ better or worse than $\mathcal{H}$ in any sense? Why?
A Better Estimate for Distinct Elements

3.1 The Problem

We revisit the DISTINCT-ELEMENTS problem from Unit 2, giving a better solution, in terms of both approximation guarantee and space usage. We also seek good time complexity. Thus, we are again in the vanilla streaming model. We have a stream \( \sigma = \langle a_1, a_2, a_3, \ldots, a_m \rangle \), with each \( a_i \in [n] \), and this implicitly defines a frequency vector \( f = (f_1, \ldots, f_n) \). Let \( d = |\{ j : f_j > 0 \}| \) be the number of distinct elements that appear in \( \sigma \). We want an \((\varepsilon, \delta)\)-approximation (as in Definition 0.2.1) to \( d \).

3.2 The BJKST Algorithm

In this section we present the algorithm dubbed BJKST, after the names of the authors: Bar-Yossef, Jayram, Kumar, Sivakumar and Trevisan [BJK'02]. The original paper in which this algorithm is presented actually gives three algorithms, the third (and, in a sense, “best”) of which we are presenting. The “zeros” notation below is the same as in Section 4.2. The values \( b \) and \( c \) are universal constants that will be determined later, based on the desired guarantees on the algorithm’s estimate.

Algorithm 4 The BJKST algorithm for DISTINCT-ELEMENTS

Initialize:
1. Choose a random hash function \( h : [n] \rightarrow [n] \) from a 2-universal family
2. Choose a random hash function \( g : [n] \rightarrow [be^{-4}\log^2 n] \) from a 2-universal family
3. \( z \leftarrow 0 \)
4. \( B \leftarrow \emptyset \)

Process (token \( j \)):
5. if zeros\((h(j)) \geq z\) then
6. \( B \leftarrow B \cup \{(g(j), \text{zeros}(h(j)))\} \)
7. while \( |B| \geq c/e^2 \) do
8. \( z \leftarrow z + 1 \)
9. shrink \( B \) by removing all \((\alpha, \beta)\) with \( \beta < z \)

Output: \( |B|/2^z \)

Intuitively, this algorithm is a refined version of the tidemark algorithm from Section 4.2. This time, rather than simply tracking the maximum value of zeros\((h(j))\) in the stream, we try to determine the size of the bucket \( B \) consisting
of all tokens \( j \) with \( \text{zeros}(h(j)) \geq z \). Of the \( d \) distinct tokens in the stream, we expect \( d/2^\epsilon \) to fall into this bucket. Therefore \( |B|2^\epsilon \) should be a good estimate for \( d \).

We want \( B \) to be small so that we can store enough information (remember, we are trying to save space) to track \( |B| \) accurately. At the same time, we want \( B \) to be large so that the estimate we produce is accurate enough. It turns out that letting \( B \) grow to about \( \Theta(1/\epsilon^2) \) in size is the right tradeoff. Finally, as a space-saving trick, the algorithm does not store the actual tokens in \( B \) but only their hash values under \( g \), together with the value of \( \text{zeros}(h(j)) \) that is needed to remove the appropriate elements from \( B \) when \( B \) must be shrunk.

We now analyze the algorithm in detail.

### 3.3 Analysis: Space Complexity

We assume that \( 1/\epsilon^2 = o(m) \): otherwise, there is no point to this algorithm! The algorithm has to store \( h, g, z, \) and \( B \). Clearly, \( h \) and \( g \) dominate the space requirement. Using the finite-field-arithmetic hash family from Exercise 2-2 for our hash functions, we see that \( h \) requires \( O(\log n) \) bits of storage. The bucket \( B \) has its size capped at \( O(1/\epsilon^2) \). Each tuple \((\alpha, \beta)\) in the bucket requires \( \log(be^{-4}\log^2 n) = O(\log(1/\epsilon) + \log \log n) \) bits to store the hash value \( \alpha \), which dominates the \( \lceil \log \log n \rceil \) bits required to store the number of zeros \( \beta \).

Overall, this leads to a space requirement of \( O(\log n + (1/\epsilon^2)(\log(1/\epsilon) + \log \log n)) \).

### 3.4 Analysis: The Quality of the Estimate

The entire analysis proceeds under the assumption that storing hash values (under \( g \)) in \( B \), instead of the tokens themselves, does not change \( |B| \). This is true whenever \( g \) does not have collisions on the set of tokens to which it is applied. By choosing the constant \( b \) large enough, we can ensure that the probability of this happening is at most \( 1/6 \), for each choice of \( h \) (you are asked to flesh this out in Exercise 3-1). Thus, making this assumption adds at most \( 1/6 \) to the error probability. We now analyze the rest of the error, under this no-collision assumption.

The basic setup is the same as in Section 2.3. For each \( j \in [n] \) and each integer \( r \geq 0 \), let \( X_{r,j} \) be an indicator random variable for the event \( \text{zeros}(h(j)) \geq r \), and let \( Y_r = \sum_{j=0}^{|T|} X_{r,j} \). Let \( T \) denote the value of \( z \) when the algorithm finishes processing the stream, and let \( \hat{d} \) denote the estimate output by the algorithm. Then we have

\[
Y_T = \text{value of } |B| \text{ when the algorithm finishes, and} \quad \hat{d} = Y_T 2^T.
\]

Proceeding as in Section 2.3, we obtain

\[
\forall r: \quad \mathbb{E}Y_r = \frac{d}{2^r}; \quad \text{Var}Y_r \leq \frac{d}{2^r}.
\tag{3.1}
\]

Notice that if \( T = 0 \), then the algorithm never incremented \( z \), which means that \( d < c/\epsilon^2 \) and \( \hat{d} = |B| = d \). In short, the algorithm computes \( d \) exactly in this case.

Otherwise \( (T \geq 1) \), we say that a \text{FAIL} event occurs if \( \hat{d} \) is not a \((1 \pm \epsilon)\)-approximation to \( d \). That is,

\[
\text{FAIL} \iff |Y_T 2^T - d| \geq \epsilon d \iff |Y_T - \frac{d}{2^T}| \geq \frac{\epsilon d}{2^T}.
\]

We can estimate this probability by summing over all possible values \( r \in \{1, 2, \ldots, \log n\} \) of \( T \). For the small values of \( r \), a failure will be unlikely when \( T = r \), because failure requires a large deviation of \( Y_r \) from its mean. For the large values of \( r \), simply having \( T = r \) is unlikely. This is the intuition for splitting the summation into two parts below. We need to choose the threshold that separates “small” values of \( r \) from “large” ones and we do it as follows.

Let \( s \) be the unique integer such that

\[
\frac{12}{\epsilon^2} \leq \frac{d}{2^r} < \frac{24}{\epsilon^2}.
\tag{3.2}
\]
Then we calculate
\[ P(\text{FAIL}) = \sum_{r=1}^{s-1} P \left\{ \left| Y_r - \frac{d}{2^r} \right| \geq \frac{\varepsilon d}{2^r} \wedge T = r \right\} \leq \sum_{r=1}^{s-1} P \left\{ \left| Y_r - \frac{d}{2^r} \right| \geq \frac{\varepsilon d}{2^r} \right\} + \sum_{r=s}^{\infty} P \{ T = r \} \]
\[ = \sum_{r=1}^{s-1} P \left\{ \left| Y_r - EY_r \right| \geq \frac{\varepsilon d}{2^r} \right\} + P \{ T \geq s \} \quad \triangleright \text{by eq. (3.1)} \]
\[ = \sum_{r=1}^{s-1} P \left\{ \left| Y_r - EY_r \right| \geq \frac{\varepsilon d}{2^r} \right\} + P \{ Y_{s-1} \geq c/\varepsilon^2 \}. \quad (3.3) \]

Bounding the terms in (3.3) using Chebyshev’s inequality and Markov’s inequality, respectively, we continue:
\[ P(\text{FAIL}) \leq \sum_{r=1}^{s-1} \frac{\text{Var} Y_r}{(\varepsilon d/2^r)^2} + \frac{E Y_{s-1}}{c/\varepsilon^2} \leq \sum_{r=1}^{s-1} \frac{2^r}{\varepsilon^2 d} + \frac{\varepsilon^2 d}{2^{2s-1}} \leq \frac{2^r}{\varepsilon^2 d} + \frac{2^2 d}{c \varepsilon^2} \leq \frac{1}{\varepsilon^2} \cdot \frac{\varepsilon^2}{12} + \frac{2 \varepsilon^2}{c} \cdot \frac{24}{\varepsilon^2} \quad \triangleright \text{by eq. (3.2)} \]
\[ \leq \frac{1}{6}, \quad (3.4) \]
where the final bound is achieved by choosing a large enough constant \( c \).

Recalling that we had started with a no-collision assumption for \( g \), the final probability of error is at most \( 1/6 + 1/6 = 1/3 \). Thus, the above algorithm \( (\varepsilon, \frac{1}{3}) \)-approximates \( d \). As before, by using the median trick, we can improve this to an \( (\varepsilon, \delta) \)-approximation for any \( 0 < \delta \leq 1/3 \), at a cost of an \( O(\log(1/\delta)) \)-factor increase in the space usage.

### 3.5 Optimality

This algorithm is very close to optimal in its space usage. Later in this course, when we study lower bounds, we shall show both an \( \Omega(\log n) \) and an \( \Omega(\varepsilon^{-2}) \) bound on the space required by an algorithm that \( (\varepsilon, \frac{1}{3}) \)-approximates the number of distinct elements. The small gap between these lower bounds and the above upper bound was subsequently closed by Kane, Nelson, and Woodruff [KNW10]: using considerably more advanced ideas, they achieved a space bound of \( O(\varepsilon^{-2} + \log n) \).

### Exercises

**3-1** Let \( \mathcal{H} \subseteq Y^X \) be a 2-universal hash family, with \( |Y| = cM^2 \), for some constant \( c > 0 \). Suppose we use a random function \( h \in_{\mathcal{H}} \) to hash a stream \( \sigma \) of elements of \( X \), and suppose that \( \sigma \) contains at most \( M \) distinct elements. Prove that the probability of a collision (i.e., the event that two distinct elements of \( \sigma \) hash to the same location) is at most \( 1/(2c) \).

**3-2** Recall that we said in class that the buffer \( B \) in the BJKST Algorithm for DISTINCT-ELEMENTS can be implemented cleverly by not directly storing the elements of the input stream in \( B \), but instead, storing the hash values of these elements under a secondary hash function whose range is of size \( cM^2 \), for a suitable \( M \).
Using the above result, flesh out the details of this clever implementation. (One important detail that you must describe: how do you implement the buffer-shrinking step?) Plug in \( c = 3 \), for a target collision probability bound of \( 1/(2c) = 1/6 \), and figure out what \( M \) should be. Compute the resulting upper bound on the space usage of the algorithm. It should work out to

\[
O \left( \log n + \frac{1}{\epsilon^2} \left( \log \frac{1}{\epsilon} + \log \log n \right) \right).
\]
Approximate Counting

While observing a data stream, we would like to know how many tokens we have seen. Just maintain a counter and increment it upon seeing a token: how simple! For a stream of length $m$, the counter would use $O(\log m)$ space which we said (in eq. 2) was the holy grail.

Now suppose, just for this one unit, that even this amount of space is too much and we would like to solve this problem using $o(\log m)$ space: maybe $m$ is really huge or maybe we have to maintain so many counters that we want each one to use less than the trivial amount of space. Since the quantity we wish to maintain—the number of tokens seen—has $m + 1$ possible values, maintaining it exactly necessarily requires $\geq \log(m + 1)$ bits. Therefore, our solution will have to be approximate.

4.1 The Problem

We are, of course, in the vanilla streaming model. In fact, the identities of the tokens are now immaterial. Therefore, we can think of the input as a stream $(1, 1, 1, \ldots)$. Our goal is to maintain an approximate count of $n$, the number of tokens so far, using $o(\log m)$ space, where $m$ is some promised upper bound on $n$. Note that we’re overriding our usual meaning of $n$: we shall do so for just this unit.

4.2 The Algorithm

The idea behind the algorithm is originally due to Morris [Sr.78], so the approximate counter provided by it is called a “Morris counter.” We now present it in a more general form, using modern notation.

Algorithm 5 The Morris counter

```
Initialize:
1: $x \leftarrow 0$

Process(token):
2: with probability $2^{-x}$ do $x \leftarrow x + 1$

Output: $2^x - 1$
```

As stated, this algorithm requires $\lceil \log m \rceil$ space after all, because the counter $x$ could grow up to $m$. However, as we shall soon see, $x$ is extremely unlikely to grow beyond $2\log m$ (say), so we can stay within a space bound of $\lceil \log \log m \rceil + O(1)$ by aborting the algorithm if $x$ does grow larger than that.
4.3 The Quality of the Estimate

The interesting part of the analysis is understanding the quality of the algorithm’s estimate. Let $C_n$ denote the (random) value of $2^x$ after $n$ tokens have been processed. Notice that $\hat{n}$, the algorithm’s estimate for $n$, equals $C_n - 1$. We shall prove that $\mathbb{E}C_n = n + 1$, which shows that $\hat{n}$ is an unbiased estimator for $n$. This by itself is not enough: we would like to prove a concentration result stating that $\hat{n}$ is unlikely to be too far from $n$. Let’s see how well we can do.

To aid the analysis, it will help to rewrite Algorithm 5 in the numerically equivalent (though space-inefficient) way shown below.

**Algorithm 6** Thought-experiment algorithm for analysis of Morris counter

**Initialize:**
1: $c \leftarrow 1$

**Process** (token):
2: with probability $1/c$ do $c \leftarrow 2c$

**Output:** $c - 1$

Clearly, the variable $c$ in this version of the algorithm is just $2^x$ in Algorithm 5, so $C_n$ equals $c$ after $n$ tokens. Here is the key lemma in the analysis.

**Lemma 4.3.1.** For all $n \geq 0$, $\mathbb{E}C_n = n + 1$ and $\text{Var}C_n = n(n - 1)/2$.

**Proof.** Let $Z_i$ be an indicator random variable for the event that $c$ is increased in line 2 upon processing the $(i+1)$th token. Then, $Z_i \sim \text{Bern}(1/C_i)$ and $C_{i+1} = (1 + Z_i)C_i$. Using the law of total expectation and the simple formula for expectation of a Bernoulli random variable,

$$
\mathbb{E}C_{i+1} = \mathbb{E}[1 + Z_i|C_i] = \mathbb{E}
\left[1 + \frac{1}{C_i}\right]C_i = 1 + \mathbb{E}C_i.
$$

Since $\mathbb{E}C_0 = 1$, this gives us $\mathbb{E}C_n = n + 1$ for all $n$. Similarly,

$$
\mathbb{E}C_i^2 = \mathbb{E}[1 + 2Z_i + Z_i^2|C_i]
= \mathbb{E}[1 + 3Z_i|C_i]C_i
= \mathbb{E}
\left[1 + \frac{3}{C_i}\right]C_i
= \mathbb{E}C_i^2 + 3(i+1).
$$

Since $\mathbb{E}C_0^2 = 1$, this gives us $\mathbb{E}C_n^2 = 1 + \sum_{i=1}^n 3i = 1 + 3n(n+1)/2$. Therefore,

$$
\text{Var}C_n = \mathbb{E}C_n^2 - (\mathbb{E}C_n)^2 = 1 + \frac{3n(n+1)}{2} - (n+1)^2 = \frac{n(n-1)}{2},
$$

upon simplification. 

Unfortunately, this variance is too large for us to directly apply the median trick introduced in Section 2.4. (Exercise: try it out! See what happens if you try to bound $\mathbb{P}\{\hat{n} < n/100\}$ using Chebyshev’s inequality and Lemma 4.3.1). However, we can still get something good out of Algorithm 5, in two different ways.

- We could tune a parameter in the algorithm, leading to a lower variance at the cost of higher space complexity. This technique, specific to this algorithm, is explored further in the exercises.
- We could plug the algorithm into a powerful, widely-applicable template that we describe below.
4.4 The Median-of-Means Improvement

As noted above, we have an estimator ($\hat{\theta}$, in this case) for our quantity of interest ($n$, in this case) that is unbiased but with variance so large that we are unable to get an ($\varepsilon, \delta$)-estimate (as in Definition 0.2.1). This situation arises often.

A generic way to deal with it is to take our original algorithm (Algorithm 5, in this case) to be a basic estimator (with variance so large that we are unable to get an $\varepsilon$-estimate) and then build from it a final estimator that runs several independent copies of the basic estimator in parallel and combines their output. The key idea is to first bring the variance down by averaging a number of independent copies of the basic estimator, and then apply the median trick.

**Lemma 4.4.1.** There is a universal positive constant $c$ such that the following holds. Let random variable $X$ be an unbiased estimator for a real quantity $Q$. Let $\{X_{ij}\}_{i \in [t], j \in [k]}$ be a collection of independent random variables with each $X_{ij}$ distributed identically to $X$, where

$$t = c \log \frac{1}{\delta}, \quad \text{and}$$

$$k = \frac{3 \Var X}{\varepsilon^2 (\E X)^2}.$$

Let $Z = \text{median}_{i \in [t]} \left( \frac{1}{k} \sum_{j=1}^{k} X_{ij} \right)$. Then, we have $\Pr\{ |Z - Q| \geq \varepsilon Q \} \leq \delta$, i.e., $Z$ is an ($\varepsilon, \delta$)-estimate for $Q$.

Thus, if an algorithm can produce $X$ using $s$ bits of space, then there is an ($\varepsilon, \delta$)-estimation algorithm using

$$O\left( s \cdot \frac{\Var X}{(\E X)^2} \cdot \frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right)$$

bits of space.

**Proof.** For each $i \in [t]$, let $Y_i = k^{-1} \sum_{j=1}^{k} X_{ij}$. Then, by linearity of expectation, we have $\E Y_i = Q$. Since the variables $X_{ij}$ are (at least) pairwise independent, we have

$$\Var Y_i = \frac{1}{k^2} \sum_{j=1}^{k} \Var X_{ij} = \frac{\Var X}{k}.$$

Applying Chebyshev’s inequality, we obtain

$$\Pr\{ |Y_i - Q| \geq \varepsilon Q \} \leq \frac{\Var Y_i}{(\varepsilon Q)^2} = \frac{\Var X}{k \E X^2} = \frac{1}{3}.$$

Now an application of a Chernoff bound (exactly as in the median trick from Section 2.4) tells us that for an appropriate choice of $c$, we have $\Pr\{ |Z - Q| \geq \varepsilon Q \} \leq \delta$. \qed

Applying the above lemma to the basic estimator given by Algorithm 5 gives us the following result.

**Theorem 4.4.2.** For a stream of length at most $m$, the problem of approximately counting the number of tokens admits an ($\varepsilon, \delta$)-estimation in $O(\log \log m \cdot \varepsilon^{-2} \log \delta^{-1})$ space.

**Proof.** If we run the basic estimator as is (according to Algorithm 5), using $s$ bits of space, the analysis in Lemma 4.3.1 together with the median-of-means improvement (Lemma 4.4.1) gives us a final ($\varepsilon, \delta$)-estimator using $O(s \varepsilon^{-2} \log \delta^{-1})$ space.

Now suppose we tweak the basic estimator to ensure $s = O(\log \log m)$ by aborting if the stored variable $x$ exceeds $2 \log m$. An abortion would imply that $C_n \geq m^2 \geq n^2$. By Lemma 4.3.1 and Markov’s inequality,

$$\Pr\{ C_n \geq n^2 \} \leq \frac{\E C_n}{n^2} = \frac{n+1}{n^2}.$$

Then, by a simple union bound, the probability that any one of the $\Theta(\varepsilon^{-2} \log \delta^{-1})$ parallel runs of the basic estimator aborts is at most $o(1)$. Thus, a final estimator based on this tweaked basic estimator produces an ($\varepsilon, \delta + o(1)$)-estimate within the desired space bound. \qed
Exercises

4-1  Here is a different way to improve the accuracy of the basic estimator in Algorithm 5. Observe that the given algorithm roughly tracks the logarithm (to base 2) of the stream length. Let us change the base from 2 to $1 + \beta$ instead, where $\beta$ is a small positive value, to be determined. Update the pseudocode using this idea, ensuring that the output value is still an unbiased estimator for $n$.

Analyze the variance of this new estimator and show that, for a suitable setting of $\beta$, it directly provides an $(\epsilon, \delta)$-estimate, using only $\log \log m + O(\log \epsilon^{-1} + \log \delta^{-1})$ bits of space.

4-2  Show how to further generalize the version of the Morris counter given by the previous exercise to solve a more general version of the approximate counting problem where the stream tokens are positive integers and a token $j$ is to be interpreted as “add $j$ to the counter.” As usual, the counter starts at zero. Provide pseudocode and rigorously analyze the algorithm’s output quality and space complexity.

4-3  Instead of using a median of means for improving the accuracy of an estimator, what if we use a mean of medians? Will it work just as well, or at all?
Finding Frequent Items via (Linear) Sketching

5.1 The Problem

We return to the FREQUENT problem that we studied in Unit 1: given a parameter $k$, we seek the set of tokens with frequency $> m/k$. The Misra–Gries algorithm, in a single pass, gave us enough information to solve FREQUENT with a second pass: namely, in one pass it computed a data structure which could be queried at any token $j \in [n]$ to obtain a sufficiently accurate estimate $\hat{f}_j$ to its frequency $f_j$. We shall now give two other one-pass algorithms for this same problem, that we can call FREQUENCY-ESTIMATION.

5.2 Sketches and Linear Sketches

Let $\text{MG}(\sigma)$ denote the data structure computed by Misra–Gries upon processing the stream $\sigma$. In $\text{??}$, we saw a procedure for combining two instances of this data structure that would let us space-efficiently compute $\text{MG}(\sigma_1 \circ \sigma_2)$ from $\text{MG}(\sigma_1)$ and $\text{MG}(\sigma_2)$, where “$\circ$” denotes concatenation of streams. Clearly, it would be desirable to be able to combine two data structures in this way, and when it can be done, such a data structure is called a sketch.

**Definition 5.2.1.** A data structure $\text{DS}(\sigma)$ computed in streaming fashion by processing a stream $\sigma$ is called a sketch if there is a space-efficient combining algorithm $\text{COMB}$ such that, for every two streams $\sigma_1$ and $\sigma_2$, we have

$$\text{COMB}(\text{DS}(\sigma_1), \text{DS}(\sigma_2)) = \text{DS}(\sigma_1 \circ \sigma_2).$$

However, the Misra–Gries has the drawback that it does not seem to extend to the turnstile (or even strict turnstile) model. In this unit, we shall design two different solutions to the FREQUENT problem that do generalize to turnstile streams. Each algorithm computes a sketch of the input stream in the above sense, but these sketches have an additional important property that we now explain.

Since algorithms for FREQUENT are computing functions of the frequency vector $f(\sigma)$ determined by $\sigma$, their sketches will naturally be functions of $f(\sigma)$. It turns out that for the two algorithms in this unit, the sketches will be linear functions. That’s special enough to call out in another definition.

**Definition 5.2.2.** A sketching algorithm “sk” is called a linear sketch if, for each stream $\sigma$ over a token universe $[n]$, $\text{sk}(\sigma)$ takes values in a vector space of dimension $\ell = \ell(n)$, and $\text{sk}(\sigma)$ is a linear function of $f(\sigma)$. In this case, $\ell$ is called the dimension of the linear sketch.

Notice that the combining algorithm for linear sketches is to simply add the sketches (in the appropriate vector space). A data stream algorithm based on a linear sketch naturally generalizes from the vanilla to the turnstile model. If the arrival of a token $j$ in the vanilla model causes us to add a vector $v_j$ to the sketch, then an update $(j, c)$ in the turnstile model is handled by adding $cv_j$ to the sketch: this handles both cases $c \geq 0$ and $c < 0$. 

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We now describe the first of our sketching algorithms, called CountSketch, which was introduced by Charikar, Chen and Farach-Colton [CCFC04]. We start with a basic sketch that already has most of the required ideas in it. This sketch takes an accuracy parameter $\varepsilon$ which should be thought of as small and positive.

Algorithm 7 CountSketch: basic estimator

Initialization:
1. $C[1 \ldots k] \leftarrow 0$, where $k := 3/\varepsilon^2$
2. Choose a random hash function $h : [n] \rightarrow [k]$ from a 2-universal family
3. Choose a random hash function $g : [n] \rightarrow \{-1, 1\}$ from a 2-universal family

Process $(token (j, c))$:  
4. $C[h(j)] \leftarrow C[h(j)] + c\, g(j)$

Output $(query a)$:  
5. report $\hat{f}_a = g(a)\, C[h(a)]$

The sketch computed by this algorithm is the array of counters $C$, which can be thought of as a vector in $\mathbb{Z}^k$. Note that for two such sketches to be combinable, they must be based on the same hash functions $h$ and $g$.

5.3 The Quality of the Basic Sketch’s Estimate

Fix an arbitrary token $a$ and consider the output $X = \hat{f}_a$ on query $a$. For each token $j \in [n]$, let $Y_j$ be the indicator for the event “$h(j) = h(a)$”. Examining the algorithm’s workings we see that a token $j$ contributes to the counter $C[h(a)]$ iff $h(j) = h(a)$, and the amount of the contribution is its frequency $f_j$ times the random sign $g(j)$. Thus,

$$X = g(a) \sum_{j=1}^{n} f_j g(j) Y_j = f_a + \sum_{j \in [n]\setminus\{a\}} f_j g(a) g(j) Y_j.$$  

Since $g$ and $h$ are independent, we have

$$\mathbb{E}[g(j) Y_j] = \mathbb{E}g(j) \cdot \mathbb{E}Y_j = 0 \cdot \mathbb{E}Y_j = 0. \quad (5.1)$$

Therefore, by linearity of expectation, we have

$$\mathbb{E}X = f_a + \sum_{j \in [n]\setminus\{a\}} f_j g(a) \mathbb{E}[g(j) Y_j] = f_a. \quad (5.2)$$

Thus, the output $X = \hat{f}_a$ is an unbiased estimator for the desired frequency $f_a$.

We still need to show that $X$ is unlikely to deviate too much from its mean. For this, we analyze its variance. By 2-universality of the family from which $h$ is drawn, we see that for each $j \in [n] \setminus \{a\}$, we have

$$\mathbb{E}Y_j^2 = \mathbb{E}Y_j = \mathbb{P}\{h(j) = h(a)\} = \frac{1}{k}. \quad (5.3)$$
Next, we use 2-universality of the family from which \( g \) is drawn, and independence of \( g \) and \( h \), to conclude that for all \( i, j \in [n] \) with \( i \neq j \), we have
\[
\mathbb{E}[g(i)g(j)Y_j] = \mathbb{E}g(i) \cdot \mathbb{E}g(j) \cdot \mathbb{E}[Y_j] = 0 \cdot 0 \cdot \mathbb{E}[Y_j] = 0.
\] (5.4)

Thus, we calculate
\[
\text{Var}X = 0 + g(a)^2 \text{Var} \left[ \sum_{j \in [n] \setminus \{a\}} f_j g(j)Y_j \right] = \mathbb{E} \left[ \sum_{j \in [n] \setminus \{a\}} f_j^2 Y_j^2 + \sum_{i,j \in [n] \setminus \{a\}} f_i f_j g(i)g(j)Y_j \right] - \left( \sum_{j \in [n] \setminus \{a\}} f_j \mathbb{E}[g(j)Y_j] \right)^2 = \sum_{j \in [n] \setminus \{a\}} f_j^2 \frac{1}{k} + 0 - 0 \quad \text{(by (5.3), (5.4), and (5.1))}
\]
\[
\Rightarrow \text{Var}X = \frac{\|f\|_2^2 - f_a^2}{k}, \quad (5.5)
\]

where \( f = f(\sigma) \) is the frequency distribution determined by \( \sigma \). From (5.2) and (5.5), using Chebyshev’s inequality, we obtain
\[
\mathbb{P} \left\{ |\hat{f}_a - f_a| \geq \epsilon \sqrt{\|f\|_2^2 - f_a^2} \right\} = \mathbb{P} \left\{ |X - \mathbb{E}X| \geq \epsilon \sqrt{\|f\|_2^2 - f_a^2} \right\} \leq \frac{\text{Var}[X]}{\epsilon^2 (\|f\|_2^2 - f_a^2)} = \frac{1}{k \epsilon^2} = \frac{1}{3}.
\]

For \( j \in [n] \), let us define \( f_{-j} \) to be the \((n - 1)\)-dimensional vector obtained by dropping the \( j \)th entry of \( f \). Then \( \|f_{-j}\|_2^2 = \|f\|_2^2 - f_j^2 \). Therefore, we can rewrite the above statement in the following more memorable form.
\[
\text{Pr} \left\{ |\hat{f}_a - f_a| \geq \epsilon \|f_{-a}\|_2 \right\} \leq \frac{1}{3}. \quad (5.6)
\]

### 5.3.2 The Final Sketch

The sketch that is commonly referred to as “Count Sketch” is in fact the sketch obtained by applying the median trick (see Section 2.4) to the above basic sketch, bringing its probability of error down to \( \delta \), for a given small \( \delta > 0 \). Thus, the Count Sketch can be visualized as a two-dimensional array of counters, with each token in the stream causing several counter updates. For the sake of completeness, we spell out this final algorithm in full below.

As in Section 2.4, a standard Chernoff bound argument proves that this estimate \( \hat{f}_a \) satisfies
\[
\mathbb{P} \left\{ |\hat{f}_a - f_a| \geq \epsilon \|f_{-a}\|_2 \right\} \leq \delta. \quad (5.7)
\]

With a suitable choice of hash family, we can store the hash functions above in \( O(t \log n) \) space. Each of the \( tk \) counters in the sketch uses \( O(\log m) \) space. This gives us an overall space bound of \( O(t \log n + tk \log m) \), which is
\[
O \left( \frac{1}{\epsilon^2} \log \frac{1}{\delta} \cdot (\log m + \log n) \right).
\]
Algorithm 8 CountSketch: final estimator

Initialize:
1: \( C[1 \ldots t][1 \ldots k] \leftarrow \vec{0}, \) where \( k := 3/\varepsilon^2 \) and \( t := O(\log(1/\delta)) \)
2: Choose \( t \) independent hash functions \( h_1, \ldots, h_t : [n] \rightarrow [k], \) each from a 2-universal family
3: Choose \( t \) independent hash functions \( g_1, \ldots, g_t : [n] \rightarrow \{-1, 1\}, \) each from a 2-universal family

Process (token \((j, c)\)):
4: for \( i \leftarrow 1 \) to \( t \) do
5: \( C[i][h_i(j)] \leftarrow C[i][h_i(j)] + cg_i(j) \)

Output (query \(a)\):
6: report \( \hat{f}_a = \text{median}_{1 \leq i \leq t} g_i(a) C[i][h_i(a)] \)

5.4 The Count-Min Sketch

Another solution to FREQUENCY-ESTIMATION is the so-called “Count-Min Sketch”, which was introduced by Cormode and Muthukrishnan [CM05]. As with the Count Sketch, this sketch too takes an accuracy parameter \( \varepsilon \) and an error probability parameter \( \delta \). And as before, the sketch consists of a two-dimensional \( t \times k \) array of counters, which are updated in a very similar manner, based on hash functions. The values of \( t \) and \( k \) are set, based on \( \varepsilon \) and \( \delta \), as shown below.

Algorithm 9 Count-Min Sketch

Initialize:
1: \( C[1 \ldots t][1 \ldots k] \leftarrow \vec{0}, \) where \( k := 2/\varepsilon \) and \( t := \lceil \log(1/\delta) \rceil \)
2: Choose \( t \) independent hash functions \( h_1, \ldots, h_t : [n] \rightarrow [k], \) each from a 2-universal family

Process (token \((j, c)\)):
3: for \( i \leftarrow 1 \) to \( t \) do
4: \( C[i][h_i(j)] \leftarrow C[i][h_i(j)] + c \)

Output (query \(a)\):
5: report \( \hat{f}_a = \min_{1 \leq i \leq t} C[i][h_i(a)] \)

Note how much simpler this algorithm is, as compared to Count Sketch! Also, note that its space usage is

\[
O\left( \frac{1}{\varepsilon} \log \frac{1}{\delta} \cdot (\log m + \log n) \right),
\]

which is better than that of Count Sketch by a \( 1/\varepsilon \) factor. The place where Count-Min Sketch is weaker is in its approximation guarantee, which we now analyze.

5.4.1 The Quality of the Algorithm’s Estimate

We focus on the case when each token \((j, c)\) in the stream satisfies \( c > 0 \), i.e., the cash register model. Clearly, in this case, every counter \( C[i][h_i(a)] \), corresponding to a token \( a \), is an overestimate of \( f_a \). Thus, we always have

\[
f_a \leq \hat{f}_a,
\]

where \( \hat{f}_a \) is the estimate of \( f_a \) output by the algorithm.

For a fixed \( a \), we now analyze the excess in one such counter, say in \( C[i][h_i(a)] \). Let the random variable \( X_i \) denote this excess. For \( j \in [n] \setminus \{a\} \), let \( Y_{i, j} \) be the indicator of the event “\( h_i(j) = h_i(a) \)”. Notice that \( j \) makes a contribution to
the counter iff $Y_{i,j} = 1$, and when it does contribute, it causes $f_j$ to be added to this counter. Thus,

$$X_i = \sum_{j \in [n] \setminus \{a\}} f_j Y_{i,j}.$$ 

By 2-universality of the family from which $h_i$ is drawn, we compute that $\mathbb{E}[Y_{i,j}] = 1/k$. Thus, by linearity of expectation,

$$\mathbb{E}[X_i] = \sum_{j \in [n] \setminus \{a\}} \frac{f_j}{k} = \frac{\|f\|_1 - f_a}{k} = \frac{\|f - f_a\|_1}{k}.$$ 

Since each $f_j \geq 0$, we have $X_i \geq 0$, and we can apply Markov’s inequality to get

$$\mathbb{P}\{X_i \geq \varepsilon \|f - a\|_1\} \leq \frac{\|f - a\|_1}{k \varepsilon \|f - a\|_1} = \frac{1}{2},$$

by our choice of $k$.

The above probability is for one counter. We have $t$ such counters, mutually independent. The excess in the output, $\hat{f}_a - f_a$, is the minimum of the excesses $X_i$, over all $i \in [t]$. Thus,

$$\mathbb{P}\{\hat{f}_a - f_a \geq \varepsilon \|f - a\|_1\} = \mathbb{P}\{\min\{X_1, \ldots, X_t\} \geq \varepsilon \|f - a\|_1\}$$

$$= \mathbb{P}\left\{ \bigwedge_{i=1}^{t} (X_i \geq \varepsilon \|f - a\|_1) \right\}$$

$$= \prod_{i=1}^{t} \mathbb{P}(X_i \geq \varepsilon \|f - a\|_1)$$

$$\leq \frac{1}{2^t},$$

and using our choice of $t$, this probability is at most $\delta$. Thus, we have shown that, with high probability,

$$f_a \leq \hat{f}_a \leq f_a + \varepsilon \|f - a\|_1,$$ 

where the left inequality always holds, and the right inequality fails with probability at most $\delta$.

The reason this estimate is weaker than that of Count Sketch is that its deviation is bounded by $\varepsilon \|f - a\|_1$, rather than $\varepsilon \|f - a\|_2$. For all vectors $z \in \mathbb{R}^n$, we have $\|z\|_1 \geq \|z\|_2$. The inequality is tight when $z$ has a single nonzero entry. It is at its weakest when all entries of $z$ are equal in absolute value: the two norms are then off by a factor of $\sqrt{n}$ from each other. Thus, the quality of the estimate of Count Sketch gets better (in comparison to Count-Min Sketch) as the stream’s frequency vector gets more “spread out”.

### 5.5 Comparison of Frequency Estimation Methods

At this point, we have studied three methods to estimate frequencies of tokens in a stream. The following table throws in a fourth method, and compares these methods by summarizing their key features.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\hat{f}_a - f_a \in \cdots$</th>
<th>Space, $O(\cdot)$</th>
<th>Error Probability</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misra–Gries</td>
<td>$[-\varepsilon |f - a|_1, 0]$</td>
<td>$\frac{1}{2} \log m + \log n$</td>
<td>$0$ (deterministic)</td>
<td>Cash register</td>
</tr>
<tr>
<td>CountSketch</td>
<td>$[-\varepsilon |f - a|_2, \varepsilon |f - a|_2]$</td>
<td>$\frac{1}{2} \log \frac{1}{\delta} \left(\log m + \log n\right)$</td>
<td>$\delta$ (overall)</td>
<td>Turnstile</td>
</tr>
<tr>
<td>Count-Min Sketch</td>
<td>$[0, \varepsilon |f - a|_1]$</td>
<td>$\frac{1}{2} \log \frac{1}{\delta} \left(\log m + \log n\right)$</td>
<td>$\delta$ (upper bound)</td>
<td>Cash register</td>
</tr>
<tr>
<td>Count/Median</td>
<td>$[-\varepsilon |f - a|_1, \varepsilon |f - a|_1]$</td>
<td>$\frac{1}{2} \log \frac{1}{\delta} \left(\log m + \log n\right)$</td>
<td>$\delta$ (overall)</td>
<td>Turnstile</td>
</tr>
</tbody>
</table>

The claims in the first row can be proved by analyzing the Misra–Gries algorithm from Lecture 1 slightly differently. The last row refers to an algorithm that maintains the same data structure as the Count-Min Sketch, but answers queries by reporting the median of the absolute values of the relevant counters, rather than the minimum. It is a simple (and instructive) exercise to analyze this algorithm and prove the claims in the last row.
Exercises

5-1 Prove that for every integer $n \geq 1$ and every vector $\mathbf{z} \in \mathbb{R}^n$, we have $\|\mathbf{z}\|_1 / \sqrt{n} \leq \|\mathbf{z}\|_2 \leq \|\mathbf{z}\|_1$. For both inequalities, determine when equality holds.

5-2 Write out the Count/Median algorithm formally and prove that it satisfies the properties claimed in the last row of the above table.

5-3 Our estimate of $\ell \|f_{-a}\|_2$ on the absolute error of CountSketch is too pessimistic in many practical situations where the data are highly skewed, i.e., where most of the weight of the vector $\mathbf{f}$ is supported on a small “constant” number of elements. To make this precise, we define $f_{-a}^{\text{res}^{(\ell)}}$ to be the $(n-1)$-dimensional vector obtained by dropping the $a$th entry of $\mathbf{f}$ and then setting the $\ell$ largest (by absolute value) entries to zero.

Now consider the CountSketch estimate $\hat{f}_a$ as computed by the algorithm in Sec 5.3.2 with the only change being that $k$ is set to $6/\varepsilon^2$. Prove that

$$\mathbb{P}\left\{ |\hat{f}_a - f_a| \geq \varepsilon \left\| f_{-a}^{\text{res}^{(\ell)}} \right\|_2 \right\} \leq \delta,$$

where $\ell = 1/\varepsilon^2$.

5-4 Consider a stream $\sigma$ in the turnstile model, defining a frequency vector $\mathbf{f} \geq \mathbf{0}$. The Count-Min Sketch solves the problem of estimating $f_j$, given $j$, but does not directly give us a quick way to identify, e.g., the set of elements with frequency greater than some threshold. Fix this.

In greater detail: Let $\alpha$ be a constant with $0 < \alpha < 1$. We would like to maintain a suitable summary of the stream (some enhanced version of Count-Min Sketch, perhaps?) so that we can, on demand, quickly produce a set $S \subseteq [n]$ satisfying the following properties w.h.p.: (1) $S$ contains every $j$ such that $f_j \geq \alpha F_1$; (2) $S$ does not contain any $j$ such that $f_j < (\alpha - \varepsilon) F_1$. Here, $F_1 = F_1(\sigma) = \|\mathbf{f}\|_1$. Design a data stream algorithm that achieves this. Your space usage, as well as the time taken to process each token and to produce the set $S$, should be polynomial in the usual parameters, $\log m$, $\log n$, and $1/\varepsilon$, and may depend arbitrarily on $\alpha$. 
Estimating Frequency Moments

6.1 Background and Motivation

We are in the vanilla streaming model. We have a stream \( \sigma = \langle a_1, \ldots, a_m \rangle \), with each \( a_j \in [n] \), and this implicitly defines a frequency vector \( f = f(\sigma) = (f_1, \ldots, f_n) \). Note that \( f_1 + \cdots + f_n = m \). The \( k \)th frequency moment of the stream, denoted \( F_k(\sigma) \) or simply \( F_k \), is defined as follows:

\[
F_k := \sum_{j=1}^{n} f_j^k = \|f\|_k^k.
\]

Using the terminology “\( k \)th” suggests that \( k \) is a positive integer. But in fact the definition above makes sense for every real \( k > 0 \). And we can even give it a meaning for \( k = 0 \), if we first rewrite the definition of \( F_k \) slightly:

\[
F_0 = \sum_{j : f_j > 0} f_j^0 = |\{ j : f_j > 0 \}|,
\]

which is the number of distinct tokens in \( \sigma \). (We could have arrived at the same result by sticking with the original definition of \( F_k \) and adopting the convention \( 0^0 = 0 \).)

We have seen that \( F_0 \) can be \((\epsilon, \delta)\)-approximated using space logarithmic in \( m \) and \( n \). And \( F_1 = m \) is trivial to compute exactly. Can we say anything for general \( F_k \)? We shall investigate this problem in this and the next few lectures.

By way of motivation, note that \( F_2 \) represents the size of the self join \( r \bowtie r \), where \( r \) is a relation in a database, with \( f_j \) denoting the frequency of the value \( j \) of the join attribute. Imagine that we are in a situation where \( r \) is a huge relation and \( n \), the size of the domain of the join attribute, is also huge; the tuples can only be accessed in streaming fashion (or perhaps it is much cheaper to access them this way than to use random access). Can we, with one pass over the relation \( r \), compute a good estimate of the self join size? Estimation of join sizes is a crucial step in database query optimization.

The solution we shall eventually see for this \( F_2 \) estimation problem will in fact allow us to estimate arbitrary equi-join sizes (not just self joins). For now, though, we give an \((\epsilon, \delta)\)-approximation for arbitrary \( F_k \), provided \( k \geq 2 \), using space sublinear in \( m \) and \( n \). The algorithm we present is due to Alon, Matias and Szegedy [AMS99], and is not the best algorithm for the problem, though it is the easiest to understand and analyze.

6.2 The (Basic) AMS Estimator for \( F_k \)

We first describe a surprisingly simple basic estimator that gets the answer right in expectation, i.e., it is an unbiased estimator. Eventually, we shall run many independent copies of this basic estimator in parallel and combine the results to get our final estimator, which will have good error guarantees.
The estimator works as follows. Pick a token from the stream $\sigma$ uniformly at random, i.e., pick a position $J \in_R [m]$. Count the length, $m$, of the stream and the number, $r$, of occurrences of our picked token $a_j$ in the stream from that point on: $r = |\{ j \geq J : a_j = a_J \}|$. The basic estimator is then defined to be $m(r^k - (r-1)^k)$.

The catch is that we don’t know $m$ beforehand, and picking a token uniformly at random requires a little cleverness, as seen in the pseudocode below.

**Algorithm 10 AMS basic estimator for frequency moments**

**Initialize:**
1. $(m, r, a) \leftarrow (0, 0, 0)$

**Process** (token $j$):
2. $m \leftarrow m + 1$
3. with probability $1/m$ do
4. $a \leftarrow j$
5. $r \leftarrow 0$
6. if $j = a$ then
7. $r \leftarrow r + 1$

**Output:** $m(r^k - (r-1)^k)$

This algorithm uses $O(\log m)$ bits to store $m$ and $r$, plus $\lceil \log n \rceil$ bits to store the token $a$, for a total space usage of $O(\log m + \log n)$. Although stated for the vanilla streaming model, it has a natural generalization to the cash register model. It is a good homework exercise to figure this out.

### 6.3 Analysis of the Basic Estimator

First of all, let us agree that the algorithm does indeed compute $r$ as described above. For the analysis, it will be convenient to think of the algorithm as picking a random token from $\sigma$ in two steps, as follows.

1. Pick a random token value, $a \in [n]$, with $P\{a = j\} = f_j/m$ for each $j \in [n]$.
2. Pick one of the $f_a$ occurrences of $a$ in $\sigma$ uniformly at random.

Let $A$ and $R$ denote the (random) values of $a$ and $r$ after the algorithm has processed $\sigma$, and let $X$ denote its output. Taking the above viewpoint, let us condition on the event $A = j$, for some particular $j \in [n]$. Under this condition, $R$ is equally likely to be any of the values $\{1, \ldots, f_j\}$, depending on which of the $f_j$ occurrences of $j$ was picked by the algorithm. Therefore,

$$
E[X \mid A = j] = E[m(R^k - (R-1)^k) \mid A = j] = \sum_{i=1}^{f_j} \frac{1}{f_j} \cdot m(i^k - (i-1)i^k) = \frac{m}{f_j} (f_j^k - 0^k).
$$

By the law of total expectation,

$$
E[X] = \sum_{j=1}^{n} P\{A = j\} E[X \mid A = j] = \sum_{j=1}^{n} \frac{f_j}{m} \cdot \frac{m}{f_j} \cdot f_j^k = F_k.
$$

This shows that $X$ is indeed an unbiased estimator for $F_k$.

We shall now bound $\text{Var}X$ from above. Calculating the expectation as before, we have

$$
\text{Var}X \leq E[X^2] = \sum_{j=1}^{n} \frac{f_j}{m} \sum_{i=1}^{f_j} \frac{1}{f_j} \cdot m^2(i^k - (i-1)i^k)^2 = m \sum_{j=1}^{n} \sum_{i=1}^{f_j} \left( f_j^k - (i-1)^k \right)^2.
$$

(6.2)
By the mean value theorem (from basic calculus), for all \( x \geq 1 \), there exists \( \xi(x) \in [x - 1, x] \) such that

\[
x^k - (x - 1)^k = k \xi(x)^{k-1} \leq k^{k-1},
\]

where the last step uses \( k \geq 1 \). Using this bound in (6.2), we get

\[
\text{Var} X \leq m \sum_{j=1}^{n} f_j \sum_{i=1}^{f_j} \left( k^{k-1} (i^{k} - (i - 1)^k) \right)
\]

\[
\leq m \sum_{j=1}^{n} k f_j^{k-1} \sum_{i=1}^{f_j} (i^{k} - (i - 1)^k)
\]

\[
= m \sum_{j=1}^{n} k f_j^{k-1} f_j^k
\]

\[
= k F_1 F_2^{k-1}.
\] (6.3)

For reasons we shall soon see, it will be convenient to bound \( \text{Var} X \) by a multiple of \( (E X)^2 \), i.e., \( F_k^2 \). To do so, we shall use the following lemma.

**Lemma 6.3.1.** Let \( n > 0 \) be an integer and let \( x_1, \ldots, x_n \geq 0 \) and \( k \geq 1 \) be reals. Then

\[
\left( \sum x_i \right) \left( \sum x_i^{2k-1} \right) \leq n^{1-1/k} \left( \sum x_i^k \right)^2,
\]

where all the summations range over \( i \in [n] \).

**Proof.** We continue to use the convention that summations range over \( i \in [n] \). Let \( x_* = \max_{i \in [n]} x_i \). Then, we have

\[
x_*^{k-1} = (x_*)^{(k-1)/k} \leq \left( \sum x_i^k \right)^{(k-1)/k}.
\] (6.4)

Since \( k \geq 1 \), by the power mean inequality (or directly, by the convexity of the function \( x \mapsto x^k \)), we have

\[
\frac{1}{n} \sum x_i \leq \left( \frac{1}{n} \sum x_i^k \right)^{1/k}.
\] (6.5)

Using (6.4) and (6.5) in the second and third steps (respectively) below, we compute

\[
\left( \sum x_i \right) \left( \sum x_i^{2k-1} \right) \leq \left( \sum x_i \right) \left( x_*^{k-1} \sum x_i^k \right)
\]

\[
\leq \left( \sum x_i \right) \left( \sum x_i^k \right)^{(k-1)/k} \left( \sum x_i^k \right)
\]

\[
\leq n^{1-1/k} \left( \sum x_i^k \right)^{1/k} \left( \sum x_i^k \right)^{(k-1)/k} \left( \sum x_i^k \right)
\]

\[
= n^{1-1/k} \left( \sum x_i^k \right)^2,
\]

which completes the proof.

Using the above lemma in (6.3), with \( x_j = f_j \), we get

\[
\text{Var} X \leq k F_1 F_2^{k-1} \leq k n^{1-1/k} F_k^2.
\] (6.6)
6.4 The Final Estimator and Space Bound

Our final $F_k$ estimator, which gives us good accuracy guarantees, is obtained by combining several independent basic estimators and using the median-of-means improvement (Lemma 4.4.1 in Section 4.4). By that lemma, we can obtain an $(\varepsilon, \delta)$-estimator for $F_k$ by combining $O(t\varepsilon^{-2}\log\delta^{-1})$ independent copies of the basic estimator $X$, where

$$t = \frac{\text{Var} X}{(\mathbb{E} X)^2} \leq \frac{kn^{1-1/k}F_k^2}{F_k^2} = kn^{1-1/k}.$$ 

As noted above, the space used to compute each copy of $X$, using Algorithm 10, is $O(\log m + \log n)$, leading to a final space bound of

$$O\left(\frac{1}{\varepsilon^2} \cdot \frac{1}{\delta} \cdot kn^{1-1/k}(\log m + \log n)\right) = \tilde{O}(\varepsilon^{-2}n^{1-1/k}). \quad (6.7)$$

6.4.1 The Soft-O Notation

For the first time in these notes, we have a space bound that is sublinear, but not polylogarithmic, in $n$ (or $m$). In such cases it is convenient to adopt an $\tilde{O}$-notation, also known as a “soft-O” notation, which suppresses factors polynomial in $\log m$, $\log n$, $\log \varepsilon^{-1}$, and $\log \delta^{-1}$. We have adopted this notation in eq. (6.7), leading to the memorable form $\tilde{O}(\varepsilon^{-2}n^{1-1/k})$. Note that we are also treating $k$ as a constant here.

The above bound is good, but not optimal, as we shall soon see. The optimal bound (upto polylogarithmic factors) is $\tilde{O}(\varepsilon^{-2}n^{1-2/k})$ instead; there are known lower bounds of $\Omega(n^{1-2/k})$ and $\Omega(\varepsilon^{-2})$. We shall see how to achieve this better upper bound in a subsequent unit.
The Tug-of-War Sketch

At this point, we have seen a sublinear-space algorithm — the AMS estimator — for estimating the $k$th frequency moment, $F_k = f_1^k + \cdots + f_n^k$, of a stream $\sigma$. This algorithm works for $k \geq 2$, and its space usage depends on $n$ as $\tilde{O}(n^{1-1/k})$. This fails to be polylogarithmic even in the important case $k = 2$, which we used as our motivating example when introducing frequency moments in the previous lecture. Also, the algorithm does not produce a sketch in the sense of Section 5.2.

But Alon, Matias and Szegedy [AMS99] also gave an amazing algorithm that does produce a sketch—a linear sketch of merely logarithmic size—which allows one to estimate $F_2$. What is amazing about the algorithm is that seems to do almost nothing.

7.1 The Basic Sketch

We describe the algorithm in the turnstile model.

Algorithm 11 Tug-of-War Sketch for $F_2$

Initialize:
1. Choose a random hash function $h : [n] \to \{-1, 1\}$ from a 4-universal family
2. $z \leftarrow 0$

Process (token $(j, c)$):
3. $z \leftarrow z + c h(j)$

Output: $z^2$

The sketch is simply the random variable $z$. It is pulled in the positive direction by those tokens $j$ that have $h(j) = 1$ and is pulled in the negative direction by the rest of the tokens; hence the name Tug-of-War Sketch. Clearly, the absolute value of $z$ never exceeds $f_1 + \cdots + f_k = m$, so it takes $O(\log m)$ bits to store this sketch. It also takes $O(\log n)$ bits to store the hash function $h$, for an appropriate 4-universal family.
7.1.1 The Quality of the Estimate

Let $Z$ denote the value of $z$ after the algorithm has processed $\sigma$. For convenience, define $Y_j = h(j)$ for each $j \in [n]$. Then $Z = \sum_{j=1}^{n} f_j Y_j$. Note that $Y_j^2 = 1$ and $\mathbb{E} Y_j = 0$, for each $j$. Therefore,

$$EZ^2 = \mathbb{E} \left[ \sum_{j=1}^{n} f_j^2 Y_j^2 + \sum_{j \neq i=1}^{n} f_i f_j Y_j \right] = \sum_{j=1}^{n} f_j^2 + \sum_{j \neq i=1}^{n} f_i f_j \mathbb{E}Y_i Y_j = F_2,$$

where we used the fact that $\{Y_j\}_{j \in [n]}$ are pairwise independent (in fact, they are 4-wise independent, because $h$ was picked from a 4-universal family). This shows that the algorithm’s output, $Z^2$, is indeed an unbiased estimator for $F_2$.

The variance of the estimator is $\text{Var} Z^2 = \mathbb{E} Z^4 - (\mathbb{E} Z^2)^2 = \mathbb{E} Z^4 - F_2^2$. We bound this as follows. By linearity of expectation, we have

$$\mathbb{E} Z^4 = \sum_{j=1}^{n} f_j^4 \mathbb{E} Y_j^4 + 6 \sum_{j=1}^{n} \sum_{i \neq j}^{n} f_i f_j f_i^2 f_j, \quad \mathbb{E} Y_j^4 = \mathbb{E} Y_i^2 Y_j = 0.$$

Suppose one of the indices in $(i, j, k, \ell)$ appears exactly once in that 4-tuple. Without loss of generality, we have $i \notin \{j, k, \ell\}$. By 4-wise independence, we then have $\mathbb{E}[Y_i Y_j Y_k Y_\ell] = \mathbb{E} Y_i \cdot \mathbb{E} [Y_j Y_k Y_\ell] = 0$, because $\mathbb{E} Y_i = 0$. It follows that the only potentially nonzero terms in the above sum correspond to those 4-tuples $(i, j, k, \ell)$ that consist either of one index occurring four times, or else two distinct indices occurring twice each. Therefore we have

$$\mathbb{E} Z^4 = \sum_{j=1}^{n} f_j^4 \mathbb{E} Y_j^4 + 6 \sum_{j=1}^{n} \sum_{i \neq j}^{n} f_i f_j f_i^2 f_j = F_4 + 6 \sum_{j=1}^{n} \sum_{i \neq j}^{n} f_i^2 f_j,$$

where the coefficient “6” corresponds to the $\binom{4}{2} = 6$ permutations of $(i, i, j, j)$ with $i \neq j$. Thus,

$$\text{Var} Z^2 = F_4 - F_2^2 + 6 \sum_{i=1}^{n} \sum_{j=i+1}^{n} f_i^2 f_j^2 = F_4 - F_2^2 + 3 \left( \sum_{j=1}^{n} f_j^2 \right)^2 - \sum_{j=1}^{n} f_j^4 = F_4 - F_2^2 + 3(F_2^2 - F_4) \leq 2F_2^2.$$

7.2 The Final Sketch

As before, having bounded the variance, we can design a final sketch from the above basic sketch by a median-of-means improvement. By Lemma 4.4.1, this will blow up the space usage by a factor of

$$\frac{\text{Var} Z^2}{(\mathbb{E} Z^2)^2} \cdot O \left( \frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right) \leq \frac{2F_2^2}{F_2^2} \cdot O \left( \frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right) = O \left( \frac{1}{\varepsilon^2} \log \frac{1}{\delta} \right)$$

in order to give an $(\varepsilon, \delta)$-estimate. Thus, we have estimated $F_2$ using space $O(\varepsilon^{-2} \log(\delta^{-1})(\log m + \log n))$, with a sketching algorithm that in fact computes a linear sketch.

7.2.1 A Geometric Interpretation

The AMS Tug-of-War Sketch has a nice geometric interpretation. Consider a final sketch that consists of $t$ independent copies of the basic sketch. Let $M \in \mathbb{R}^{t \times n}$ be the matrix that “transforms” the frequency vector $f$ into the $t$-dimensional sketch vector $y$. Note that $M$ is not a fixed matrix but a random matrix with ±1 entries: it is drawn from a certain distribution described implicitly by the hash family. Specifically, if $M_{ij}$ denotes the $(i, j)$-entry of $M$, then $M_{ij} = h_i(j)$, where $h_i$ is the hash function used by the $i$th basic sketch.
Let \( t = 6/\varepsilon^2 \). By stopping the analysis in Lemma 4.4.1 after the Chebyshev step (and before the “median trick” Chernoff step), we obtain that

\[
\Pr \left\{ \left| \frac{1}{t} \sum_{i=1}^{t} y_i^2 - F_2 \right| \geq \varepsilon F_2 \right\} \leq \frac{1}{3},
\]

where the probability is taken with respect to the above distribution of \( M \), resulting in a random sketch vector \( y = (y_1, \ldots, y_t) \). Thus, with probability at least 2/3, we have

\[
\left\| \frac{1}{\sqrt{t}} M f \right\|_2 = \frac{1}{\sqrt{t}} \| y \|_2 \in \left[ \sqrt{1-\varepsilon} \| f \|_2, \sqrt{1+\varepsilon} \| f \|_2 \right] \subseteq \left[ (1-\varepsilon) \| f \|_2, (1+\varepsilon) \| f \|_2 \right].
\] (7.1)

This can be interpreted as follows. The (random) matrix \( M/\sqrt{t} \) performs a dimension reduction, simplifying an \( n \)-dimensional vector \( f \) to a \( t \)-dimensional sketch \( y \)—with \( t = O(1/\varepsilon^2) \)—while preserving \( \ell_2 \)-norm within a \( (1 \pm \varepsilon) \) factor. Of course, this is only guaranteed to happen with probability at least 2/3. But clearly this correctness probability can be boosted to an arbitrary constant less than 1, while keeping \( t = O(1/\varepsilon^2) \).

The “amazing” AMS sketch now feels quite natural, under this geometric interpretation. We are using dimension reduction to maintain a low-dimensional image of the frequency vector. This image, by design, has the property that its \( \ell_2 \)-length approximates that of the frequency vector very well. Which of course is what we’re after, because the second frequency moment, \( F_2 \), is just the square of the \( \ell_2 \)-length.

Since the sketch is linear, we now also have an algorithm to estimate the \( \ell_2 \)-difference \( \| f(\sigma) - f(\sigma') \|_2 \) between two streams \( \sigma \) and \( \sigma' \).

**Exercises**

7-1 In Section 6.1, we noted that \( F_2 \) represents the size of a self join in a relational database and remarked that our \( F_2 \) estimation algorithm would allow us to estimate arbitrary equi-join sizes (not just self joins). Justify this by designing a sketch that can scan a relation in one streaming pass such that, based on the sketches of two different relations, we can estimate the size of their join. Explain how to compute the estimate.

Recall that for two relations (i.e., tables in a database) \( r(A, B) \) and \( s(A, C) \), with a common attribute (i.e., column) \( A \), we define the join \( r \Join s \) to be a relation consisting of all tuples \((a, b, c)\) such that \((a, b) \in r \) and \((a, c) \in s \). Therefore, if \( f_{r,j} \) and \( f_{s,j} \) denote the frequencies of \( j \) in the first columns (i.e., “\( A \”)-columns) of \( r \) and \( s \), respectively, and \( j \) can take values in \([n]\), then the size of the join is \( \sum_{j=1}^{n} f_{r,j} f_{s,j} \).

7-2 As noted in Section 7.2.1, the \( t \times n \) matrix that realizes the tug-of-war sketch has every entry in \( \{-1/\sqrt{t}, 1/\sqrt{t} \} \); in particular, every entry is nonzero. Therefore, in a streaming setting, updating the sketch in response to a token arrival takes \( \Theta(t) \) time, under the reasonable assumption that the processor can perform arithmetic on \( \Theta(\log n) \)-bit integers in constant time.

Consider the matrix \( P \in \mathbb{R}^{t \times n} \) given by

\[
P_{ij} = \begin{cases} g(i), & \text{if } i = h(j), \\ 0, & \text{otherwise}, \end{cases}
\]

where hash functions \( g: [n] \rightarrow \{-1, 1\} \) and \( h: [n] \rightarrow [t] \) are drawn from a \( k_1 \)-universal and a \( k_2 \)-universal family, respectively. Show that using \( P \) as a sketch matrix (for some choice of constants \( k_1 \) and \( k_2 \)) leads to dimension reduction guarantees similar to eq. (7.1). What is the per-token update time achievable using \( P \) as the sketch matrix?
Estimating Norms Using Stable Distributions

As noted at the end of Unit 7, the AMS Tug-of-War sketch allows us to estimate the $\ell_2$-difference between two data streams. Estimating similarity metrics between streams is an important class of problems, so it is nice to have such a clean solution for this specific metric.

However, this raises a burning question: Can we do the same for other $\ell_p$ norms, especially the $\ell_1$ norm? The $\ell_1$-difference between two streams can be interpreted (modulo appropriate scaling) as the total variation distance (a.k.a., statistical distance) between two probability distributions: a fundamental and important metric. Unfortunately, although our log-space $F_2$ algorithm automatically gave us a log-space $\ell_2$ algorithm, the trivial log-space $F_1$ algorithm works only in the cash register model and does not give an $\ell_1$ algorithm at all.

It turns out that thinking harder about the geometric interpretation of the AMS Tug-of-War Sketch leads us on a path to polylogarithmic space $\ell_p$ norm estimation algorithms, for all $p \in (0, 2]$. Such algorithms were given by Indyk [Ind06], and we shall study them now. For the first time in this course, it will be necessary to gloss over several technical details of the algorithms, so as to have a clear picture of the important ideas.

### 8.1 A Different $\ell_2$ Algorithm

The length-preserving dimension reduction achieved by the Tug-of-War Sketch is reminiscent of the famous Johnson-Lindenstrauss Lemma [JL84, FM88]. One high-level way of stating the JL Lemma is that the random linear map given by a $t \times n$ matrix whose entries are independently drawn from the standard normal distribution $N(0, 1)$ is length-preserving (up to a scaling factor) with high probability. To achieve $1 \pm \varepsilon$ error, it suffices to take $t = O(1/\varepsilon^2)$. Let us call such a matrix a JL Sketch matrix. Notice that the sketch matrix for the Tug-of-War sketch is a very similar object, except that

1. its entries are uniformly distributed in $\{-1, 1\}$: a much simpler distribution;
2. its entries do not have to be fully independent: 4-wise independence in each row suffices; and
3. it has a succinct description: it suffices to describe the hash functions that generate the rows.

The above properties make the Tug-of-War Sketch “data stream friendly”. But as a thought experiment one can consider an algorithm that uses a JL Sketch matrix instead. It would give a correct algorithm for $\ell_2$ estimation, except that its space usage would be very large, as we would have to store the entire sketch matrix. In fact, since this hypothetical algorithm calls for arithmetic with real numbers, it is unimplementable as stated.

Nevertheless, this algorithm has something to teach us, and will generalize to give (admittedly unimplementable) $\ell_p$ algorithms for each $p \in (0, 2]$. Later we shall make these algorithms realistic and space-efficient. For now, we consider the basic sketch version of this algorithm, i.e., we maintain just one entry of $\mathbf{Mf}$, where $\mathbf{M}$ is a JL Sketch matrix. The pseudocode below shows the operations involved.
Algorithm 12 Sketch for $\ell_2$ based on normal distribution

Initialize:
1. Choose $Y_1, \ldots, Y_n$ independently, each from $\mathcal{N}(0, 1)$
2. $z \leftarrow 0$

Process (token $(j, c)$):
3. $z \leftarrow z + cY_j$

Output: $z^2$

Let $Z$ denote the value of $x$ when this algorithm finishes processing $\sigma$. Then $Z = \sum_{j=1}^n f_j Y_j$. From basic statistics, using the independence of the collection $\{Y_j\}_{j \in [n]}$, we know that $Z$ has the same distribution as $\|f\|_2 Y$, where $Y \sim \mathcal{N}(0, 1)$. This is a fundamental property of the normal distribution.\(^1\) Therefore, we have $\mathbb{E} Z^2 = \|f\|_2^2 = F_2$, which gives us our unbiased estimator for $F_2$.

### 8.2 Stable Distributions

The fundamental property of the normal distribution that was used above has a generalization, which is the key to generalizing this algorithm. The next definition captures the general property.

**Definition 8.2.1.** Let $p > 0$ be a real number. A probability distribution $\mathcal{D}_p$ over the reals is said to be $p$-stable if for all integers $n \geq 1$ and all $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, the following property holds. If $X_1, \ldots, X_n$ are independent and each $X_i \sim \mathcal{D}_p$, then $c_1X_1 + \cdots + c_nX_n$ has the same distribution as $\bar{c}X$, where $X \sim \mathcal{D}_p$ and

$$
\bar{c} = (c_1^p + \cdots + c_n^p)^{1/p} = \|c\|_p.
$$

The concept of stable distributions dates back to Lévy [Lév54] and is more general than what we need here. It is known that $p$-stable distributions exist for all $p \in (0, 2]$, and do not exist for any $p > 2$. The fundamental property above can be stated simply as: “The standard normal distribution is 2-stable.”

Another important example of a stable distribution is the Cauchy distribution, which can be shown to be 1-stable. Just as the standard normal distribution has density function

$$
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},
$$

the Cauchy distribution also has a density function expressible in closed form as

$$
c(x) = \frac{1}{\pi(1 + x^2)}.
$$

But what is really important to us is not so much that the density function of $\mathcal{D}_p$ be expressible in closed form, but that it be easy to generate random samples drawn from $\mathcal{D}_p$. The Chambers-Mallows-Stuck method [CMS76] gives us the following simple algorithm. Let

$$
X = \frac{\sin(p \theta)}{(\cos \theta)^{1/p}} \left( \frac{\cos((1 - p) \theta)}{\ln(1/r)} \right)^{(1-p)/p},
$$

where $(\theta, r) \in \mathbb{R} \times [\pi/2, \pi/2] \times [0, 1]$. Then the distribution of $X$ is $p$-stable.

Replacing $\mathcal{N}(0, 1)$ with $\mathcal{D}_p$ in the above pseudocode, where $\mathcal{D}_p$ is $p$-stable, allows us to generate a random variable distributed according to $\mathcal{D}_p$ “scaled” by $\|f\|_p$. Note that the scaling factor $\|f\|_p$ is the quantity we want to estimate. To estimate it, we shall simply take the median of a number of samples from the scaled distribution, i.e., we shall maintain a sketch consisting of several copies of the basic sketch and output the median of (the absolute values of) the entries of the sketch vector. Here is our final “idealized” sketch.

\(^1\)The proof of this fact is a nice exercise in calculus.
Algorithm 13 Indyk’s sketch for $\ell_p$ estimation

Initialize:
1: $M[1\ldots t][1\ldots n] \leftarrow tn$ independent samples from $\mathcal{D}_p$, where $t = O(\epsilon^{-2} \log(\delta^{-1}))$
2: $z[1\ldots t] \leftarrow \tilde{0}$

Process (token $(j, c)$):
3: for $i = 1$ to $t$ do
4: \hfill $z[i] \leftarrow z[i] + cM[j][j]$

Output: $\text{median}_{1\leq i\leq t} |z_i| / \text{median}(|\mathcal{D}_p|)$

8.3 The Median of a Distribution and its Estimation

To analyze this algorithm, we need the concept of the median of a probability distribution over the reals. Let $\mathcal{D}$ be an absolutely continuous distribution, let $\phi$ be its density function, and let $X \sim \mathcal{D}$. A median of $\mathcal{D}$ is a real number $\mu$ that satisfies
\[
\frac{1}{2} = \mathbb{P}\{X \leq \mu\} = \int_{-\infty}^{\mu} \phi(x) \, dx.
\]

The distributions that we are concerned with here are nice enough to have uniquely defined medians; we will simply speak of the median of a distribution. For such a distribution $\mathcal{D}$, we will denote this unique median as $\text{median}(\mathcal{D})$.

For a distribution $\mathcal{D}$, with density function $\phi$, we denote by $|\mathcal{D}|$ the distribution of the absolute value of a random variable drawn from $\mathcal{D}$. It is easy to show that the density function of $|\mathcal{D}|$ is $\psi$, where
\[
\psi(x) = \begin{cases} 2\phi(x), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}
\]

For $p \in (0, 2]$ and $c \in \mathbb{R}$, let $\phi_{p,c}$ denote the density function of the distribution of $c|X|$, where $X \sim \mathcal{D}_p$, and let $\mu_{p,c}$ denote the median of this distribution. Note that
\[
\phi_{p,c}(x) = \frac{1}{c} \phi_{p,1}\left(\frac{x}{c}\right), \quad \text{and} \quad \mu_{p,c} = c\mu_{p,1}.
\]

Let $Z_i$ denote the final value of $z_i$ after Algorithm 13 has processed $\sigma$. By the earlier discussion, and the definition of $p$-stability, we see that $Z_i = \|f\|_p Z$, where $Z \sim \mathcal{D}_p$. Therefore, $|Z_i| / \text{median}(|\mathcal{D}_p|)$ has a distribution whose density function is $\phi_{p,\lambda}$, where $\lambda = \|f\|_p / \text{median}(|\mathcal{D}_p|) = \|f\|_p / \mu_{p,1}$. Thus, the median of this distribution is $\mu_{p,\lambda} = \lambda \mu_{p,1} = \|f\|_p$.

The algorithm—which seeks to estimate $\|f\|_p$—can thus be seen as attempting to estimate the median of an appropriate distribution by drawing $t = O(\epsilon^{-2} \log \delta^{-1})$ samples from it and outputting the sample median. We now show that this does give a fairly accurate estimate.

8.4 The Accuracy of the Estimate

Lemma 8.4.1. Let $\epsilon > 0$, and let $\mathcal{D}$ be a distribution over $\mathbb{R}$ with density function $\phi$, and with a unique median $\mu > 0$. Suppose that $\phi$ is absolutely continuous on $[(1-\epsilon)\mu, (1+\epsilon)\mu]$ and let $\phi_* = \min\{\phi(z) : z \in [(1-\epsilon)\mu, (1+\epsilon)\mu]\}$. Let $Y = \text{median}_{1\leq i\leq t} Z_i$, where $Z_1, \ldots, Z_t$ are independent samples from $\mathcal{D}$. Then
\[
\mathbb{P}\{|Y - \mu| \geq \epsilon \mu\} \leq 2 \exp\left(-\frac{2}{3} \epsilon^2 \mu^2 t^2 \right).
\]

Proof. We bound $\mathbb{P}\{Y < (1-\epsilon)\mu\}$ from above. A similar argument bounds $\mathbb{P}\{Y > (1+\epsilon)\mu\}$ and to complete the proof we just add the two bounds.
Let $\Phi(y) = \int_{-\infty}^{y} \phi(z) \, dz$ be the cumulative distribution function of $\mathcal{D}$. Then, for each $i \in [t]$, we have

$$\mathbb{P}\{Z_i < (1 - \varepsilon)\mu\} = \int_{-\infty}^{\mu} \phi(z) \, dz - \int_{(1 - \varepsilon)\mu}^{\mu} \phi(z) \, dz$$

$$= \frac{1}{2} - \Phi(\mu) + \Phi((1 - \varepsilon)\mu)$$

$$= \frac{1}{2} - \varepsilon \mu \phi(\xi),$$

for some $\xi \in [(1 - \varepsilon)\mu, \mu]$, where the last step uses the mean value theorem and the fundamental theorem of calculus: $\Phi' = \phi$. Let $\alpha$ be defined by

$$\left(\frac{1}{2} - \varepsilon \mu \phi(\xi)\right)(1 + \alpha) = \frac{1}{2}. \quad (8.1)$$

Let $N = |\{i \in [t] : Z_i < (1 - \varepsilon)\mu\}|$. By linearity of expectation, we have $\mathbb{E}N = (\frac{1}{2} - \varepsilon \mu \phi(\xi))t$. If the sample median, $Y$, falls below a limit $\lambda$, then at least half the $Z_i$'s must fall below $\lambda$. Therefore

$$\mathbb{P}\{Y < (1 - \varepsilon)\mu\} \leq \mathbb{P}\{N \geq t/2\} = \mathbb{P}\{N \geq (1 + \alpha)\mathbb{E}N\} \leq \exp(-\mathbb{E}N\alpha^2/3),$$

by a standard Chernoff bound. Now, from (8.1), we derive $\mathbb{E}N\alpha = \varepsilon \mu \phi(\xi)t$ and $\alpha \geq 2\varepsilon \mu \phi(\xi)$. Therefore

$$\mathbb{P}\{Y < (1 - \varepsilon)\mu\} \leq \exp\left(-\frac{2}{3}\varepsilon^2\mu^2\phi^2(\xi)^2t\right) \leq \exp\left(-\frac{2}{3}\varepsilon^2\mu^2\phi^2(\xi)^2t\right).$$

To apply the above lemma to our situation we need an estimate for $\phi$. We will be using the lemma with $\phi = \phi_{p,\lambda}$ and $\mu = \mu_{p,\lambda} = \|f\|_p / \mu_{p,1}$. Therefore,

$$\mu_{p,\lambda} \phi_{p,\lambda} = \mu_{p,\lambda} \min\{\phi_{p,\lambda}(z) : z \in [(1 - \varepsilon)\mu_{p,\lambda}, (1 + \varepsilon)\mu_{p,\lambda}]\}$$

$$= \lambda \mu_{p,1} \min\left\{\frac{1}{\lambda} \phi_{p,1}\left(\frac{\xi}{\lambda}\right) : z \in [(1 - \varepsilon)\lambda \mu_{p,1}, (1 + \varepsilon)\lambda \mu_{p,1}]\right\}$$

$$= \mu_{p,1} \cdot \min\{\phi_{p,1}(y) : y \in [(1 - \varepsilon)\mu_{p,1}, (1 + \varepsilon)\mu_{p,1}]\},$$

which is a constant depending only on $p$: call it $c_p$. Thus, by Lemma 8.4.1, the output $Y$ of the algorithm satisfies

$$\mathbb{P}\left\{|Y - \|f\|_p | \geq \varepsilon \|f\|_p\right\} \leq \exp\left(-\frac{2}{3}c_p^2\delta^2\right) \leq \delta,$$

for the setting $t = (3/(2c_p^2))e^{-2\log(\delta^{-1})}$.

**8.5 Annoying Technical Details**

There are two glaring issues with the “idealized” sketch we have just discussed and proven correct. As stated, we do not have a proper algorithm to implement the sketch, because

- the sketch uses real numbers, and algorithms can only do bounded-precision arithmetic; and
- the sketch depends on a huge matrix — with $n$ columns — that does not have a convenient implicit representation.

We will not go into the details of how these matters are resolved, but here is an outline.

We can approximate all real numbers involved by rational numbers with sufficient precision, while affecting the output by only a small amount. The number of bits required per entry of the matrix $M$ is only logarithmic in $n$, $1/\varepsilon$ and $1/\delta$.

We can avoid storing the matrix $M$ explicitly by using a pseudorandom generator (PRG) designed to work with space-bounded algorithms. One such generator is given by a theorem of Nisan [Nis90]. Upon reading an update to a token $j$, we use the PRG (seeded with $j$ plus the initial random seed) to generate the $j$th column of $M$. This transformation blows up the space usage by a factor logarithmic in $n$ and adds $1/n$ to the error probability.
Sparse Recovery

We have seen several algorithms that maintain a small linear sketch of an evolving vector $f$ (that undergoes turnstile updates) and then answer questions about $f$ using this sketch alone. When $f$ is arbitrary, such sketches must necessarily throw away most of the information in $f$ due to size constraints. However, when $f$ itself holds only a small amount of information—for instance, by having at most $s$ many nonzero coordinates—it is reasonable to ask whether such a linear sketch can allow a complete reconstruction of $f$ on demand. This is the problem of sparse recovery.

Sparse recovery is an important area of research first studied extensively in the world of signal processing (especially for the problem of image acquisition), where it also goes by such names as compressed sensing and compressive sensing. It has many connections to topics throughout computer science and mathematics [GI10]. We will focus on a clean version of the problem, rather than the most general version.

### 9.1 The Problem

We are in the turnstile streaming model. Stream tokens are of the form $(j, c)$, where $j \in [n]$ and $c \in \mathbb{Z}$ and have the usual meaning “$f_j \leftarrow f_j + c$.” The net result is to build a frequency vector $f \in \mathbb{Z}^n$. We will need a bound on how large the individual entries of $f$ can get: we assume that $\|f\|_\infty := \max_{j \in [n]} |f_j| \leq M$ at all times. One often assumes that $M = \text{poly}(n)$, so that $\log M = \Theta(\log n)$.

The support of a vector $v = (v_1, \ldots, v_n)$ is defined to be the set of indices where the vector is nonzero:

$$\text{supp } v = \{j \in [n] : v_j \neq 0\}.$$  

We say that $v$ is $s$-sparse if $|\text{supp } v| \leq s$, and the cardinality $|\text{supp } v|$ is called the sparsity of $v$.

The streaming $s$-sparse recovery problem is to maintain a sketch $y$ of the vector $f$, as the stream comes in, so that if $f$ ends up being $s$-sparse, it can be recovered from $y$, and otherwise we can detect the non-sparsity. Think $s \ll n$. We would like our sketch size to be “not much bigger” than $s$, with the dependence on the ambient dimension $n$ being only polylogarithmic.

### 9.2 Special case: $1$-sparse recovery

An important special case is $s = 1$, i.e., the problem of $1$-sparse recovery. Let us first consider a promise version of the problem, where it is guaranteed that the final frequency vector $f$ resulting from the stream is indeed $1$-sparse.

For this, consider the following very simple deterministic algorithm: maintain the population size $\ell$ (the net number
of tokens seen) and the frequency-weighted sum $z$ of all token values seen. In other words,

$$\ell = \sum_{j=1}^{n} f_j; \quad z = \sum_{j=1}^{n} jf_j.$$  

Since $f$ is 1-sparse, $f = \lambda e_i$ for some $i \in [n]$ and $\lambda \in \mathbb{Z}$, where $e_i$ is the standard basis vector that has a 1 entry at index $i$ and 0 entries elsewhere. Therefore, $\ell = \lambda$ and $z = if_i = i\lambda = i\ell$. It follows that we can recover $f$ using

$$f = \begin{cases} 0, & \text{if } \ell = 0, \\ \ell e_{z/\ell}, & \text{otherwise.} \end{cases}$$

Of course, in the absence of the promise, the above technique cannot detect whether $f$ is indeed 1-sparse.

To solve the more general problem of 1-sparse detection and recovery, we employ the important idea of fingerprinting. Informally speaking, a fingerprint is a randomized mapping from a large object to a much smaller sketch with the property that two distinct large objects will very likely have distinct sketches, just as two distinct humans will very likely have distinct fingerprints. In computer science, an often-used technique is to treat the large object as a polynomial and use a random evaluation of that polynomial as its fingerprint.

These ideas are incorporated into the algorithm below, which is parametrized by a finite field $\mathbb{F}$, the size of which affects the eventual error probability. As we shall see, a good choice for $\mathbb{F}$ is one with $n^3 < |\mathbb{F}| \leq 2n^3$. We know that we can always find a finite field (in fact a prime field) with size in this range.

**Algorithm 14** Streaming 1-sparse detection and recovery

**Initialize:**

1. $(\ell, z, p) \leftarrow (0, 0, 0)$ \hspace{1cm} $\triangleright$ (population, sum, fingerprint)
2. $r \leftarrow$ uniform random element of finite field $\mathbb{F}$ \hspace{1cm} $\triangleright$ $|\mathbb{F}|$ controls the error probability

**Process** (token $(j, c)$):

3. $\ell \leftarrow \ell + c$
4. $z \leftarrow z + cj$
5. $p \leftarrow p + cr^{j}$

**Output:**

6. if $\ell = z = p = 0$ then declare $f = 0$ \hspace{1cm} $\triangleright$ very likely correct
7. else if $z/\ell \notin [n]$ then declare $\|f\|_0 > 1$ \hspace{1cm} $\triangleright$ definitely correct
8. else if $p \neq \ell r^{z/\ell}$ then declare $\|f\|_0 > 1$ \hspace{1cm} $\triangleright$ definitely correct
9. else
10. declare $f = \ell e_{z/\ell}$ \hspace{1cm} $\triangleright$ very likely correct

### 9.3 Analysis: Correctness, Space, and Time

Let $R$ be the random value of $r$ picked in the initialization. The final values of $\ell$, $z$, and $p$ computed by Algorithm 14 are

$$\ell = \sum_{j=1}^{n} f_j, \quad (9.1)$$

$$z = \sum_{j=1}^{n} jf_j, \quad (9.2)$$

$$p = \sum_{j=1}^{n} f_j R^j = q(R), \quad (9.3)$$
where \( q(X) \in \mathbb{F}[X] \) is the polynomial \( q(X) = \sum_{i=1}^{n} f_i X^i \). This polynomial \( q(X) \) is naturally in 1-to-1 correspondence with the vector \( f \): coefficients of the polynomial correspond to entries in the vector.

Consider the case when \( f \) is indeed 1-sparse: say \( f = \lambda e_i \). As before, we have \( \ell = \lambda \) and \( z = i \ell \). So, \( z/\ell = i \in [n] \) and \( p = f_i R^i = \ell R^i \). In the subcase \( f = 0 \), this makes \( p = 0 \) and the algorithm gives a correct output, by the logic of line 6. In the subcase \( ||f||_0 = 1 \), the algorithm reaches line 10 and thus gives a correct output. Overall, we see that there are no false negatives.

Consider the case when \( f \) is not 1-sparse. Then, for certain “bad” choices of \( R \), Algorithm 14 might err by giving a false positive. To be precise, an error happens when \( \ell = z = p = 0 \) in line 6 or \( z/\ell \) happens to lie in \([n]\) and then \( p = \ell R^i \) in line 8. These cases can be “cleverly” combined by defining

\[
i = \begin{cases} 
0, & \text{if } \ell = \lambda = 0 \text{ or } z/\ell \notin [n], \\
\frac{z}{\ell}, & \text{otherwise}.
\end{cases}
\]

A false positive happens only if \( q(R) - \ell R^i = 0 \), i.e., only if \( R \) is a root of the polynomial \( q(X) - \ell X^i \). This latter polynomial has degree at most \( n \) and is not the zero polynomial, since \( q(X) \) has at least two nonzero coefficients. Recall the following basis theorem from algebra.

**Theorem 9.3.1.** Over any field, a nonzero polynomial of degree \( d \) has at most \( d \) roots. \( \square \)

By Theorem 9.3.1, \( q(X) - \ell X^i \) has at most \( n \) roots. Since \( R \) is drawn uniformly from \( \mathbb{F} \),

\[
\mathbb{P}[\text{false positive}] \leq \frac{n}{|\mathbb{F}|} = O\left( \frac{1}{n^2} \right), \tag{9.4}
\]

by choosing \( \mathbb{F} \) such that \( n^3 < |\mathbb{F}| < 2n^3 \).

For the space and time cost analysis, suppose that \( |\mathbb{F}| = \Theta(n^3) \). Algorithm 14 maintains the values \( \ell, z, p, \) and \( r \). Recall that \( ||f||_\infty \leq M \) at all times. By eqs. (9.1) and (9.2), \( ||f||_\infty \leq nM \) and \( |z| \leq n^2 M \); so \( \ell \) and \( z \) take up at most \( O(\log n + \log M) \) bits of space. Since \( p \) and \( r \) lie in \( \mathbb{F} \), they take up at most \( |\log |\mathbb{F}|| = O(\log n) \) bits of space. Overall, the space usage is \( O(\log n + \log M) \), which is \( \tilde{O}(1) \) under the standard (and reasonable) assumption that \( M = \text{poly}(n) \).

Assume, further, that the processor has a word size of \( \Theta(\log n) \). Then, by using repeated squaring for the computation of \( r^t \), the processing of each token takes \( O(\log n) \) arithmetic operations and so does the post-processing to produce the output. We have thus proved the following result.

**Theorem 9.3.2.** Given turnstile updates to a vector \( f \in \mathbb{Z}^n \) where \( ||f||_\infty = \text{poly}(n) \), the 1-sparse detection and recovery problem for \( f \) can be solved with error probability \( O(1/n) \) using a linear sketch of size \( O(1) \). The sketch can be updated in \( \tilde{O}(1) \) time per token and the final output can be computed in \( \tilde{O}(1) \) time from the sketch. \( \square \)

## 9.4 General Case: \( s \)-sparse Recovery

We turn to the general case. Let \( s \ll n \) be a parameter. We would like to recover the vector \( f \) exactly, provided \( ||f||_0 \leq s \), and detect that \( ||f||_0 > s \) if not. Our solution will make use of the above 1-sparse recovery sketch as a black-box data structure, denoted \( \mathcal{D} \). When an instance of \( \mathcal{D} \) produces an actual 1-sparse vector as output, let’s say that it reports positive; otherwise, let’s say that it reports negative. Remember that there are no false negatives and that a false positive occurs with probability \( O(1/n) \).

The basic idea behind the algorithm is to hash the universe \([n]\) down to the range \([2s]\), thereby splitting the input stream into \(2s\) sub-streams, each consisting of items that hash to a particular target. This range size is large enough to single out any particular item in \( \text{supp} f \), causing the sub-stream containing that item to have a 1-sparse frequency vector. By repeating this idea some number of times, using independent hashes, we get the desired outcome. The precise logic is given in Algorithm 15 below.

Clearly, the space usage of this algorithm is dominated by the \( 2st \) copies of \( \mathcal{D} \). Plugging in the value of \( t \) and our previous space bound on \( \mathcal{D} \), the overall space usage is

\[
O(s(\log s + \log \delta^{-1})(\log n + \log M)).
\]
Algorithm 15 Streaming $s$-sparse recovery

Initialize:
1: $t \leftarrow \lceil \log(s/\delta) \rceil$ \hfill $\triangleright$ $\delta$ controls the failure probability
2: $D[1 \ldots t][1 \ldots 2s] \leftarrow \emptyset$ \hfill $\triangleright$ each $D[i][k]$ is a copy of $\mathcal{D}$
3: Choose independent hash functions $h_1, \ldots, h_t : [n] \to [2s]$, each from a 2-universal family

Process (token $(j, c)$):
4: foreach $i \in [t]$ do
5: update $D[i][h_i(j)]$ with token $(j, c)$

Output:
6: $A \leftarrow$ (empty associative array)
7: foreach $i \in [t], k \in [2s]$ do \hfill $\triangleright$ query all copies of $\mathcal{D}$ and collate outputs
8: if $D[i][k]$ reports positive and outputs $\lambda e_a$ where $\lambda \neq 0$ then
9: if $a \in \text{keys}(A)$ and $A[a] \neq \lambda$ then abort
10: $A[a] \leftarrow \lambda$
11: if $|\text{keys}(A)| > s$ then abort
12: declare $f = \sum_{a \in \text{keys}(A)} A[a] e_a$

Turning to the error analysis, suppose that $f$ is indeed $s$-sparse and that $\text{supp} f \subseteq \{j_1, \ldots, j_t\}$. Notice that the data structure $D[i][k]$ sketched the sub-stream consisting of those tokens $(j, c)$ for which $h_i(j) = k$. This sub-stream is 1-sparse if there is at most one $j \in \text{supp} f$ such that $h_i(j) = k$. Therefore, Algorithm 15 correctly outputs $f$ if both of the following happen.

$[\text{SR}_1]$ For each $j \in \text{supp} f$, there exists $i \in [t]$ such that $h_i^{-1}(h_i(j)) \cap \text{supp} f = \{j\}$.

$[\text{SR}_2]$ None of the $2st$ copies of $\mathcal{D}$ gives a false positive.

Consider a particular item $j \in \text{supp} f$. For each $i \in [t]$,

$$
\Pr \{h_i^{-1}(h_i(j)) \cap \text{supp} f \neq \{j\}\} = \Pr \{\exists j' \in \text{supp} f: j' \neq j \land h_i(j') = h_i(j)\}
\leq \sum_{j' \in \text{supp} f \setminus \{j\}} \Pr \{h_i(j') = h_i(j)\}
\leq \sum_{j' \in \text{supp} f \setminus \{j\}} \frac{1}{2s} \leq \frac{1}{2},
$$

where the penultimate step uses the 2-universality property. By the mutual independence of the $t$ hash functions,

$$
\Pr \{\text{SR}_1 \text{ fails for item } j\} = \prod_{i=1}^{t} \Pr \{h_i^{-1}(h_i(j)) \cap \text{supp} f \neq \{j\}\} \leq \left(\frac{1}{2}\right)^t \leq \frac{\delta}{s}.
$$

By a union bound over the items in $\text{supp} f$,

$$
\Pr(\neg \text{SR}_1) \leq |\text{supp} f| \cdot \frac{\delta}{s} \leq \delta.
$$

By another union bound over the copies of $\mathcal{D}$, using the false positive rate bound from eq. (9.4),

$$
\Pr(\neg \text{SR}_2) \leq 2st \cdot O \left(\frac{1}{n^2}\right) = o(1),
$$

since $s \leq n$. Overall, the probability that the recovery fails is at most $\delta + o(1)$. 

\[\text{PAGE}\ 43\]
Theorem 9.4.1. Given turnstile updates to a vector \( f \in \mathbb{Z}^n \) where \( \|f\|_\infty = \text{poly}(n) \), the \( s \)-sparse recovery problem for \( f \) can be solved with error probability \( \delta \) using a linear sketch of size \( O(s \log(s/\delta) \log n) \).

With just a little more work (and analysis), we can solve the \( s \)-sparse detection and recovery problem as well. This is left as an exercise.

Exercises

9-1 Suppose that in Algorithm 14, we modified line 6 to declare \( f = 0 \) whenever \( \ell = z = 0 \), without using the value of \( p \). Would this still be correct? Why?

9-2 As described and analyzed, Algorithm 15 requires the vector \( f \) to be \( s \)-sparse. Explain clearly why it does not already solve the problem of detecting whether this \( s \)-sparsity condition holds. Make a simple enhancement to the algorithm to enable this detection and analyze your modified algorithm to prove that it works.
Weight-Based Sampling

The problem of sampling a coordinate at random from a high-dimensional vector is important both in its own right and as a key primitive for other, more advanced, data stream algorithms [?]. For instance, consider a stream of visits by customers to the busy website of some business or organization. An analyst might want to sample one or a few customers according to some distribution defined by the frequencies of various customers’ visits. Letting $f \in \mathbb{Z}^n$ be the vector of frequencies (as usual), we would like to pick a coordinate $i \in [n]$ according to some distribution $\pi = (\pi_1, \ldots, \pi_n) \in [0,1]^n$. Here are three specific examples of distributions we may want to sample from.

- Sample a customer uniformly from the set of all distinct customers who visited the website. In our formalism, $\pi_i = 1/\|f\|_0$ for each $i \in \text{supp} f$ and $\pi_i = 0$ for all other $i \in [n]$. This is called $\ell_0$-sampling.
- Sample a customer with probability proportional to their visit frequency. In our formalism, $\pi_i = |f_i|/\|f\|_1$. This is called $\ell_1$-sampling.
- Sample a customer with frequent visitor weighted disproportionately highly, for instance, proportional to the square of their visit frequency. In our formalism, $\pi_i = |f_i|^2/\|f\|_2^2$. More generally, one can consider $\pi_i = |f_i|^p/\|f\|_p^p$, for a parameter $p > 0$. This is called $\ell_p$-sampling.

### 10.1 The Problem

We shall study an algorithm for approximate $\ell_0$ sampling and then one for approximate $\ell_2$-sampling: these algorithms are “approximate” because the distribution over items (coordinates) that they induce are close to, but not identical to, the respective target distributions.

To be precise, the task is as follows. The input is a turnstile stream $\sigma$ that builds up a frequency vector $f \in \mathbb{Z}^n$. Assume that at all times we have $\|f\|_\infty \leq M$, for some bound $M = \text{poly}(n)$. We must maintain a small-sized linear sketch of $f$ from which we can produce a random index $i \in [n]$ distributed approximately according to $\pi$, where $\pi$ is one of the distributions listed above. With small probability, we may output $\text{FAIL}$ instead of producing an index in $[n]$. If the output $I$ of the algorithm satisfies

$$\mathbb{P}\{I = i\} \in [(1 - \varepsilon)\pi_i - \theta, (1 + \varepsilon)\pi_i + \theta], \text{ for each } i \in [n], \quad (10.1)$$

$$\mathbb{P}\{I = \text{FAIL}\} \leq \delta, \quad (10.2)$$

where $\pi$ is the distribution corresponding to $\ell_p$-sampling, then we say that the algorithm is an $(\varepsilon, \theta, \delta)$-$\ell_p$-sampler.

Monemizadeh and Woodruff [?] gave sampling algorithms that exploited this relaxation of producing the distribution $\pi$ only approximately. More recently, Jayaram and Woodruff [?] showed that space-efficient perfect sampling—where there is no approximation, but a small failure probability is allowed—is possible.
10.2 The $\ell_0$-Sampling Problem

Our task is to design an $(\epsilon, \theta, \delta)$-$\ell_0$-sampler as defined in eqs. (10.1) and (10.2), with $\pi$ defined by

$$
\pi_i = \frac{1}{\|f\|_0}, \quad \text{if } i \in \text{supp } f,
$$

$$
\pi_i = 0, \quad \text{otherwise}.
$$

It is promised that $f \neq 0$, so that this problem is well-defined.

A key observation is that, thanks to the sparse recovery sketches designed in Unit 9, this problem becomes simple if $f$ is promised to be $s$-sparse, for a small $s$. This leads to the following idea: choose a random subset $S \subseteq [n]$ of coordinates so that if we form a sub-vector of $f$ by zeroing out all coordinates outside $S$, the sub-vector is likely to be very sparse. If we filter the input stream $\sigma$ and only retain tokens that update items in $S$, we can simulate the stream that builds up this sub-vector and feed this simulated stream to a sparse recovery data structure.

Let $d := \|f\|_0$. If $S$ is formed by picking each coordinate in $[n]$ with probability about $1/d$, independently, then the sub-vector described above has a good—i.e., $\Omega(1)$—chance of being 1-sparse. Of course, we don’t know $d$ in advance, so we have to prepare for several possibilities by using several sets $S$, using several different coordinate retention probabilities.

10.2.1 An Idealized Algorithm

It will be useful to assume, without loss of generality, that $n$ is a power of 2.

We realize the idea outlined above by using hash functions to create the random sets: the set of items in $[n]$ that are hashed to a particular location form one of the sets $S$ mentioned above. We begin with an idealized algorithm that uses uniformly random hash functions on the domain $[n]$. Of course, storing such a hash function requires $\Omega(n)$ bits, so this does not achieve sublinear storage. We can address this issue in one of the following ways.

- We can apply Nisan’s space-bounded PRG to generate these hashes pseudorandomly, in limited space.
- We can use $k$-universal hash families, for a suitable not-too-large $k$, which is more space efficient.

Either of these approaches will introduce a small additional error in the algorithm’s output.

The algorithm below uses several copies of a 1-sparse detection and recovery data structure, $\mathcal{D}$. We can plug in the data structure developed in Unit 9, which uses $O(\log n + \log M)$ space and has a false positive probability in $O(1/n^2)$.

Algorithm 16 Idealized $\ell_0$-sampling algorithm using full randomness

Initialize:

1. for $\ell \leftarrow 0$ to $\log n$ do
2. choose $h_\ell : [n] \to \{0, 1\}^\ell$ uniformly at random from all such functions
3. $\mathcal{D}_\ell \leftarrow \emptyset$  \quad \triangleright 1-sparse detection and recovery data structure

Process (token $(j, c)$):

4. for $\ell \leftarrow 0$ to $\log n$ do
5. if $h_\ell(j) = 0$ then \quad \triangleright happens with probability $2^{-\ell}$
6. feed $(j, c)$ to $\mathcal{D}_\ell$

Output:

7. for $\ell \leftarrow 0$ to $\log n$ do
8. if $\mathcal{D}_\ell$ reports positive and outputs $\lambda e_i$ with $\lambda \neq 0$ then
9. yield $(i, \lambda)$ and stop
10. output FAIL

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10.2.2 The Quality of the Output

Let \( d = |\text{supp}(f)| \). Consider level \( \ell \) such that \( \frac{1}{4d} \leq \frac{1}{2^\ell} < \frac{1}{2} \).

\[
\mathbb{P}\{\text{vector fed to } D_\ell \text{ is 1-
\text{sparse}}\} = \Pr \left[ \bigvee_{j \in \text{supp}(f)} \left( h_\ell(j) = 0 \right) \land \bigwedge_{i \in \text{supp}(f): i \neq j} \left( h_\ell(i) \neq 0 \right) \right]
\]

\[
= \sum_{j \in \text{supp}(f)} \Pr \left( h_\ell(j) = 0 \right) \land \bigwedge_{i \in \text{supp}(f): i \neq j} \left( h_\ell(i) \neq 0 \right)
\]

\[
= \sum_{j \in \text{supp}(f)} \Pr \{ h_\ell(j) = 0 \} \Pr \left[ \bigwedge_{i \in \text{supp}(f): i \neq j} \left( h_\ell(i) \neq 0 \right) \bigg| (h_\ell(j) = 0) \right]
\]

\[
\geq \sum_{j \in \text{supp}(f)} \frac{1}{2^\ell} \left( 1 - \Pr \left[ \bigvee_{i \in \text{supp}(f): i \neq j} \left( h_\ell(i) = 0 \right) \bigg| (h_\ell(j) = 0) \right] \right)
\]

\[
= \sum_{j \in \text{supp}(f)} \frac{1}{2^\ell} \left( 1 - \sum_{i \in \text{supp}(f): i \neq j} \Pr \{ (h_\ell(i) = 0) \bigg| (h_\ell(j) = 0) \} \right)
\]

\[
\geq \frac{d}{2^\ell} \left( 1 - \frac{d - 1}{2^\ell} \right) \geq \frac{d}{2^\ell} \left( 1 - \frac{d}{2^\ell} \right) \geq \frac{1}{4} \left( 1 - \frac{1}{2} \right) = \frac{1}{8}.
\]

To achieve error probability of \( O(\delta) \), the number of 1-
\text{sparse detection and recovery data structures to be used is\( O(\log n \log \frac{1}{\delta}) \), each of which fails with probability \( O(\frac{1}{n}) \). So

\[
\mathbb{P}\{\text{some data structure fails}\} \leq O \left( \frac{\log n \log \frac{1}{\delta}}{n} \right).
\]

The rest of the notes for this unit are VERY rough and I am not yet decided on how much of this material I will actually teach. When they have been polished/finalized, this notice will be removed.

A practical algorithm.

Algorithm 17 \( \ell_0 \)-sampling

**Initialize:**

1: \( h_\ell : [n] \to \{0, 1\}^\ell \) for \( \ell \in \{0, 1, \ldots, \log n\} \)
2: \( D_0, D_1, \ldots, D_{\log n} \)

\( \triangleright 2 \)-universal, independent

\( \triangleright 1 \)-sparse detection and recovery data structures

**Process (token \((j, c)\)):**

3: for \( 0 \leq \ell \leq \log n \) do
4: if \( h_\ell(j) = 0 \) then
5: feed \((j, c)\) to \( D_\ell \)

**Output:**

6: for \( 0 \leq \ell \leq \log n \) do
7: if \( D_\ell \) reports positive and outputs \( \lambda e_\ell \) then
8: yield \((i, \lambda)\) and stop
9: output “FAIL”

To argue that the element returned by the algorithm is indeed uniformly random in the support of \( f \), we need to use \( O(\log n) \)-universal hash families in the algorithm. A \( O(\log n) \)-universal hash family is also minwise independent.
A hash family \( \mathcal{H} \) of functions \( h : [n] \rightarrow [n] \) is minwise independent if \( \forall X \subseteq [n] \forall x \in [n] \setminus X \) we have
\[
\Pr_{h \in \mathcal{H}} [h(x) < \min_{y \in X} h(y)] = \frac{1}{|X| + 1}.
\]

10.3 \( \ell_2 \) Sampling

Given a turnstile stream \( \sigma \) generating a frequency vector \( f = (f_1, f_2, \ldots, f_n) \), we seek to output \((J, \hat{f}_j^2)\) such that
\[
\Pr[J = j] \approx \frac{f_j^2}{\|f\|^2} \quad \text{and} \quad \hat{f}_j^2 \text{ is an approximation for } f_j^2.
\]
The algorithm that we discuss here is known as the “precision sampling algorithm”, which was introduced by Andoni, Krauthgamer, and Onak. In this algorithm we use count sketch data structure to estimate the modified frequency of elements in \([n]\). For each element \( j \in [n] \), we define its modified frequency \( g_j = f_j \sqrt{u_j} \) where \( u_j \) is a random number chosen uniformly from the interval \([\frac{1}{n^2}, 1]\). Assume \( \varepsilon > 0 \) be an accuracy parameter for the algorithm (\( \varepsilon \) should be thought of as very small).

10.3.1 An \( \ell_2 \)-sampling Algorithm

**Algorithm 18** \( \ell_2 \)-sampling or precision sampling

1. Initialize:
   1. \( u_1, u_2, \ldots, u_n \in R \left[ \frac{1}{n^2}, 1 \right] \).
2. Count sketch data structure \( D \) for \( g = (g_1, g_2, \ldots, g_n) \).
3. Process \((j, c)\):
   1. \( \text{feed} \left( j, \frac{c}{\sqrt{u_j}} \right) \text{into} \ D \).
4. Output:
   1. for each \( j \in [n] \) do
      1. \( \hat{g}_j \leftarrow \text{estimate of } g_j \text{ from } D \).
      2. \( \hat{f}_j \leftarrow \hat{g}_j \sqrt{u_j} \).
      3. \( X_j \leftarrow \begin{cases} 1 & \text{if } \hat{g}_j^2 = \hat{f}_j^2 \geq \frac{4}{\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \)
   8. if \( \exists \text{ unique } j \text{ with } X_j = 1 \) then
      9. output \((j, \hat{f}_j^2)\).
   else
   10. FAIL.

Remarks:

- As written we need to store \( u = \{u_1, u_2, \ldots, u_n\} \).
- But in fact \( u \) is just random and input independent, and the procedure is space-bounded. So we can use Nisan’s PRG to reduce random seed.
- As written \( \Pr[FAIL] \approx 1 - \varepsilon \), but just repeat \( \Theta \left( \frac{1}{\varepsilon} \log \frac{1}{\delta} \right) \) times for \( \Pr[Fail] \leq \delta \).
- As written, algorithm 18 will work as analysed, assuming \( 1 \leq F_2 \leq 2 \).

10.3.2 Analysis:

Let \( F_2 = \sum_{j=1}^{n} f_j^2 = \|f\|^2 \) and \( F_2(g) = \sum_{j=1}^{n} g_j^2 \).
Claim 1. \( E F_2(g) \leq 5 \log n \).

Proof.

\[
E F_2(g) = \sum_{j=1}^{n} E g_j^2
\]

\[
= \sum_{j=1}^{n} E f_j^2 u_j
\]

\[
= \sum_{j=1}^{n} f_j^2 E \frac{1}{u_j}
\]

\[
= F_2 \int_{1/n^2}^{1} \frac{1}{u} \cdot \frac{1}{1 - 1/n^2} \quad \text{since } E u_i = E u_j \forall i, j,
\]

\[
= F_2 \ln n^2
\]

\[
\leq (4 + \Theta(\frac{1}{n^2})) \cdot \ln n
\]

\[
\leq 5 \log n
\]

Consider estimate \( \hat{g}_j \) derived from a \( 1 \times k \) count sketch data structure: \( \hat{g}_j = g_j + Z_j \) where \( Z_j \) is the sum of contribution from \( i \neq j \) that collide with \( j \). Recall that \( E Z_j = 0 \), \( E Z_j^2 = \frac{1}{k} \sum_{i \neq j} g_i^2 \leq \frac{F_2(g)}{k} \). Hence, by applying Markov inequality, we get \( \Pr[Z_j^2 \geq \frac{3F_2(g)}{k}] \leq \frac{1}{3} \). Now consider the following two cases:

- If \( |g_j| \geq \frac{2}{\varepsilon} \), then \( \hat{g}_j^2 = |g_j + Z_j|^2 = e^{\pm \varepsilon} g_j^2 \).
- Else \( |g_j| < \frac{2}{\varepsilon} \). Then,

\[
|\hat{g}_j^2 - g_j^2| \leq (|g_j| + |Z_j|)^2 - |g_j|^2,
\]

\[
= |Z_j|^2 + 2|g_j Z_j|,
\]

\[
\leq Z_j^2 (1 + \frac{4}{\varepsilon}).
\]

So with probability at least \( \frac{2}{3} \),

\[
|\hat{g}_j^2 - g_j^2| \leq \frac{3F_2(g)}{k} \cdot \frac{\varepsilon + 4}{\varepsilon}
\]

since \( \Pr[Z_j^2 \geq \frac{3F_2(g)}{k}] \leq \frac{1}{3} \).

We pick \( k = \frac{680 \log n}{\varepsilon} = \Theta(\frac{\log n}{\varepsilon}) \). Using Claim 1, we get \( \Pr[F_2(g) \geq 50 \log n] \leq \frac{5 \log n}{\varepsilon \log n} = \frac{1}{10} \). Hence with probability \( 1 - (1/3 + 1/10) \),

\[
|\hat{g}_j^2 - g_j^2| \leq 1.
\]

So in both the cases, \( \hat{g}_j^2 = e^{\pm \varepsilon} g_j^2 \pm 1, \Rightarrow \hat{f}_j^2 = e^{\pm \varepsilon} f_j^2 \pm u_j \). We observe that when \( X_j = 1, u_j \leq \frac{\varepsilon f_j^2}{4} \). So \( \hat{f}_j^2 = e^{\pm \varepsilon} f_j^2 \pm \frac{\varepsilon f_j^2}{4} = e^{\pm \varepsilon} f_j^2 \pm \varepsilon \hat{f}_j^2 \). Using the fact that \( 1 + \varepsilon \approx e^\varepsilon \) for \( \varepsilon \approx 0 \), we get after rearranging, \( \hat{f}_j^2 = e^{\pm 2 \varepsilon} f_j^2 \).
Now we are ready to lower bound the probability that algorithm 18 produces an output:

\[ \Pr[j \text{ is output}] = \Pr \left[ X_j = 1 \land \left( \bigwedge_{i \neq j} X_i = 0 \right) \right], \]

\[ = \Pr[X_j = 1] \cdot \left( 1 - \Pr \left[ \bigvee_{i \neq j} (X_i = 1) \mid X_j = 0 \right] \right), \]

\[ = \Pr[X_j = 1] \cdot \left( 1 - \Pr \left[ \bigvee_{i \neq j} (X_i = 1) \right] \right) \text{ assuming } X_i \text{s are pairwise independent}, \]

\[ \geq \Pr[X_j = 1] \cdot \left( 1 - \sum_{i \neq j} \Pr[X_i = 1] \right) \text{ by union bound}, \]

\[ = \Pr \left[ u_j \leq \frac{\varepsilon f_j^2}{4} \right] \cdot \left( 1 - \sum_{i \neq j} \Pr \left[ u_i \leq \frac{\varepsilon f_i^2}{4} \right] \right), \]

\[ \approx \frac{\varepsilon f_j^2}{4} \cdot \left( 1 - \sum_{i \neq j} \frac{\varepsilon f_i^2}{4} \right) \text{ pretending } u_j \in [0, 1], \]

\[ \geq \frac{\varepsilon f_j^2}{4} \cdot \left( 1 - \sum_{i \neq j} \frac{\varepsilon f_i^2}{4} \right), \]

\[ = \frac{\varepsilon e^{\pm 2\varepsilon} f_j^2}{4} \cdot \left( 1 - \sum_{i} \frac{\varepsilon e^{2\varepsilon} f_i^2}{4} \right), \]

\[ \geq \frac{\varepsilon e^{-2\varepsilon} f_j^2}{4} \cdot \left( 1 - \frac{\varepsilon e^{2\varepsilon}}{2} \right) \text{ using } f_2 \leq 2. \]

Now we upper bound the probability that algorithm 18 produces an output:

\[ \Pr[j \text{ is output}] \leq \Pr\{X_j = 1\}, \]

\[ \approx \frac{\varepsilon f_j^2}{4} \text{ pretending } u_j \in [0, 1], \]

\[ \leq \frac{\varepsilon e^{2\varepsilon} f_j^2}{4}, \]

\[ = \frac{\varepsilon^{2\varepsilon}}{4} \cdot \varepsilon f_j^2. \]

Hence, \( \Pr\{j \text{ is output}\} = \frac{1}{4} e^{\pm 3\varepsilon} \cdot \varepsilon f_j^2. \) So success probability of algorithm 18 is given by:

\[ \Pr\{-\text{FAIL}\} = \sum_j \Pr[j \text{ is output}] = \frac{1}{4} e^{\pm 3\varepsilon} \cdot \varepsilon f_2. \]

Finally, the sampling probability of \( j \) is given by:

\[ \Pr\{\text{sample } j \mid \text{-FAIL}\} = \frac{f_j^2 \cdot e^{\pm 6\varepsilon}}{f_2}. \]
Finding the Median

Finding the mean of a stream of numbers is trivial, requiring only logarithmic space. How about finding the median? To understand this, we shall now study the main algorithm from a seminal paper of Munro and Paterson [MP80] that has come to be recognized as the “first data streaming paper.”

11.1 The Problem

We are working in the vanilla stream model. Our input is a stream of numbers \( \sigma = \langle a_1, a_2, \ldots, a_m \rangle \), with each \( a_i \in [n] \). Our goal is to output the median of these numbers.

Recall that the median of the collection \( \{a_i\}_{i \in [m]} \) is defined as the real number \( y \) that minimizes \( \sum_{i=1}^{m} |y - a_i| \), the total distance to the numbers in the collection. Let \( \pi \) be a permutation of \([m]\) that sorts the collection, i.e., makes \( a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(m)} \). It is easy to see that if \( m \) is odd then \( y = a_{\pi(\lceil m/2 \rceil)} \), whereas if \( m \) is even then \( y = \frac{1}{2}(a_{\pi(m/2)} + a_{\pi(m/2+1)}) \). To avoid annoying case analysis depending on the parity of \( m \), which is irrelevant from an algorithmic point of view, we shall redefine the median of the collection to be \( a_{\pi(\lceil m/2 \rceil)} \) always.

In fact we shall consider—and solve—a more general problem, namely SELECTION. Here, we are given a specific integer \( r \) and asked to find the \( r \)th smallest element in the collection \( \{a_i\}_{i \in [m]} \), i.e., the number \( a_{\pi(r)} \).

Suppose that the stream is sorted. Then the median of the stream is the value in the middle position (when \( m \) is odd) or the average of the two on either side of the middle (when \( m \) is even). More concisely, the median is the average of the values at the \( \lceil n/2 \rceil \) and \( \lfloor n/2 \rfloor \) indices. (Running two copies of our algorithm finds the mathematical median.)

This leads us to a slight generalization: finding the \( r \)th element in a sorted set.

**Definition 11.1.1.** Given a set \( S \) of elements from a domain \( D \) with an imposed order and an element \( x \in D \), the rank of \( x \) is the count of elements in the set \( S \) that are not greater than \( x \).

\[
\text{rank}(x, S) = |\{y \in S : y \leq x\}|
\]

This definition of rank makes sense even if \( x \) does not belong to \( S \). Note: we are working with sets (no repeats), so assume they are all distinct. Otherwise, we’d have to consider the rank of an element in a multiset.

* * * In progress * * *

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This page contains the text related to Unit 11, focusing on the problem of finding the median in a stream of numbers. The text explains the definition of median, the problem of finding the median in a stream, and introduces the SELECTION problem as a generalization. It also defines the rank of an element in a set or multiset.
12.1 Extent Measures and Minimum Enclosing Ball

Up to this point, we have been interested in statistical computations. Our data streams have simply been “collections of tokens.” We did not care about any structure amongst the tokens, because we were only concerned with functions of the token frequencies. The median and selection problems introduced some slight structure: the tokens were assumed to have a natural ordering.

Streams represent data sets. The data points in many data sets do have natural structure. In particular, many data sets are naturally geometric: they can be thought of as points in $\mathbb{R}^d$, for some dimension $d$. This motivates a host of computational geometry problems on data streams. We shall now study a simple problem of this sort, which is in fact a representative of a broader class of problems: namely, the estimation of extent measures of point sets.

Broadly speaking, an extent measure estimates the “spread” of a collection of data points, in terms of the size of the smallest object of a particular shape that contains the collection. Here are some examples.

- Minimum bounding box (two variants: axis-parallel or not)
- Minimum enclosing ball, or MEB
- Minimum enclosing shell (a shell being the difference of two concentric balls)
- Minimum width (i.e., min distance between two parallel hyperplanes that sandwich the data)

For this lecture, we focus on the MEB problem. Our solution will introduce the key concept of a coreset, which forms the basis of numerous approximation algorithms in computational geometry, including data stream algorithms for all of the above problems.

12.2 Coresets and Their Properties

The idea of a coreset was first formulated in Agarwal, Har-Peled and Varadarajan [AHPV04]; the term “coreset” was not used in that work, but became popular later. The same authors have a survey [AHPV05] that is a great reference on the subject, with full historical context and a plethora of applications. Our exposition will be somewhat specialized to target the problems we are interested in.

We define a “cost function” $C$ to be a family of functions $\{C_P\}$ parametrized by point sets $P \subseteq \mathbb{R}^d$. For each $P$, we have a corresponding function $C_P : \mathbb{R}^d \rightarrow \mathbb{R}_+$. We say that $C$ is monotone if

$$\forall P \subseteq \mathbb{R}^d \forall x \in \mathbb{R}^d : C_Q(x) \leq C_P(x).$$
We are given a stream $\sigma$ of points in $\mathbb{R}^d$, and we wish to compute the minimum value of the corresponding cost function $C_\sigma$. To be precise, we want to estimate $\inf_{x \in \mathbb{R}^d} C_\sigma(x)$; this will be our extent measure for $\sigma$.

The minimum enclosing ball (or MEB) problem consists of finding the minimum of the cost function $C_\sigma$, where

$$C_\sigma(x) := \max_{y \in \sigma} \|x - y\|_2,$$

i.e., the radius of the smallest ball centered at $x$ that encloses the points in $\sigma$.

**Definition 12.2.1.** Fix a real number $\alpha \geq 1$, and a cost function $C$ parameterized by point sets in $\mathbb{R}^d$. We say $Q$ is an $\alpha$-coreset for $P \subseteq \mathbb{R}^d$ (with respect to $C$) if $Q \subseteq P$, and

$$\forall T \subseteq \mathbb{R}^d \forall x \in \mathbb{R}^d : C_{Q\cup T}(x) \leq C_{P\cup T}(x) \leq \alpha C_{Q\cup T}(x).$$

**Merge Property** If $Q$ is an $\alpha$-coreset for $P$ and $Q'$ is a $\beta$-coreset for $P'$, then $Q \cup Q'$ is an $(\alpha \beta)$-coreset for $P \cup P'$.

**Reduce Property** If $Q$ is an $\alpha$-coreset for $P$ and $R$ is a $\beta$-coreset for $Q$, then $R$ is an $(\alpha \beta)$-coreset for $P$.

**Disjoint Union Property** If $Q, Q'$ are $\alpha$-coresets for $P, P'$ respectively, and $P \cap P' = \emptyset$, then $Q \cup Q'$ is an $\alpha$-coreset for $P \cup P'$.

To repeat: every coreset satisfies the merge and reduce properties. The proof is left as an easy exercise.

### 12.3 A Coreset for MEB

For nonzero vectors $u, v \in \mathbb{R}^d$, let $\text{ang}(u, v)$ denote the angle between them, i.e.,

$$\text{ang}(u, v) := \arccos \frac{\langle u, v \rangle}{\|u\|_2 \|v\|_2},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. We shall call a collection of vectors $\{u_1, \ldots, u_t\} \subseteq \mathbb{R}^d \setminus \{0\}$ a $\theta$-grid if, for every nonzero vector $x \in \mathbb{R}^d$, there is a $j \in [t]$ such that $\text{ang}(x, u_j) \leq \theta$. We think of $\theta$ as being “close to zero.”

The following geometric theorem is well known; it is obvious for $d = 2$, but requires a careful proof for higher $d$.

**Theorem 12.3.1.** In $\mathbb{R}^d$, there is a $\theta$-grid consisting of $O(1/\theta^{d-1})$ vectors. In particular, for $\mathbb{R}^2$, this bound is $O(1/\theta)$.

Using this, we can construct a small coreset for MEB.

**Theorem 12.3.2.** In $d \geq 2$ dimensions, the MEB cost function admits a $(1 + \varepsilon)$-coreset of size $O(1/\varepsilon(d-1)/2)$.

**Proof.** Let $\{u_1, \ldots, u_t\}$ be a $\theta$-grid in $\mathbb{R}^d$, for a parameter $\theta = \theta(\varepsilon)$ to be chosen later. By Theorem 12.3.1, we may take $t = O(1/\theta^{d-1})$. Our proposed coreset for a point set $P \subseteq \mathbb{R}^d$ shall consist of the two extreme points of $P$ along each of the directions $u_1, \ldots, u_t$. To be precise, let $P$ be given. We then define

$$Q := \bigcup_{i=1}^{t} \left\{ \arg \max_{x \in P} \langle x, v_i \rangle, \arg \min_{x \in P} \langle x, v_i \rangle \right\}.$$
We claim that $Q$ is a $(1 + \varepsilon)$-coreset of $P$, for a suitable choice of $\theta$.

Since the MEB cost function is monotone, the left inequality in (12.2) always holds. We prove the right inequality “by picture” (see below); making this rigorous is left as an exercise.

Take an arbitrary $x \in \mathbb{R}^d$ and $T \subseteq \mathbb{R}^d$, and let $z$ be the farthest point from $x$ in the set $P \cup T$. If $z \in T$, then

$$C_{Q,T}(x) \geq \|x - z\|_2 = C_{P,T}(x).$$

Otherwise, we have $z \in P$. By the grid property, there is a direction $u_i$ that makes an angle of at most $\theta$ with $\vec{xz}$. Let $y$ be the point such that $\vec{xy}$ is parallel to $u_i$ and $\|x - y\|_2 = \|x - z\|_2$, and let $z'$ be the orthogonal projection of $z$ onto $\vec{xy}$. Then, $Q$ contains a point from $P$ whose orthogonal projection on $\vec{xy}$ lies to the right of $z'$ (by construction of $Q$) and to the left of $y$ (because $z$ is farthest). Therefore

$$C_{Q,T}(x) \geq C_Q(x) \geq \|x - z'\|_2 \geq \|x - z\| \cos \theta = C_{P,T}(x) \cos \theta.$$ 

Using $\sec \theta \leq 1 + \theta^2$ (which holds for $\theta$ small enough), we obtain $C_{P,T}(x) \leq (1 + \theta^2)C_{Q,T}(x)$. Since this holds for all $x \in \mathbb{R}^d$, the right inequality in (12.2) holds with $\alpha = 1 + \theta^2$. Since we wanted $Q$ to be a $(1 + \varepsilon)$-coreset, we may take $\theta = \sqrt{\varepsilon}$.

Finally, with this setting of $\theta$, we have $|Q| \leq 2t = O(1/\varepsilon^{(d-1)/2})$, which completes the proof.

### 12.4 Data Stream Algorithm for Coreset Construction

We now turn to algorithms. Fix a monotone cost function $C$, for point sets in $\mathbb{R}^d$, and consider the “$C$-minimization problem,” i.e., the problem of estimating $\inf_{x \in \mathbb{R}^d} C(x)$.

**Theorem 12.4.1.** Suppose $C$ admits $(1 + \varepsilon)$-coresets of size $A(\varepsilon)$, and that these coresets have the disjoint union property. Then the $C$-minimization problem has a data stream algorithm that uses space $O(A(\varepsilon/\log m) \cdot \log m)$ and returns a $(1 + \varepsilon)$-approximation.

**Proof.** Our algorithm builds up a coreset for the input stream $\sigma$ recursively from coresets for smaller and smaller substreams of $\sigma$ in a way that is strongly reminiscent of the Munro-Paterson algorithm for finding the median.
Set \( \delta = \frac{\varepsilon}{\log m} \). We run a number of streaming algorithms in parallel, one at each “level”: we denote the level-\( j \) algorithm by \( \mathcal{A}_j \). By design, \( \mathcal{A}_j \) creates a virtual stream that is fed into \( \mathcal{A}_{j+1} \). Algorithm \( \mathcal{A}_0 \) reads \( \sigma \) itself, placing each incoming point into a buffer of large enough size \( B \). When this buffer is full, it computes a \((1 + \delta)\)-coreset of the points in the buffer, sends this coreset to \( \mathcal{A}_1 \), and empties the buffer.

For \( j \geq 1 \), \( \mathcal{A}_j \) receives a coreset at a time from \( \mathcal{A}_{j-1} \). It maintains up to two such coresets. Whenever it has two of them, say \( P \) and \( P' \), it computes a \((1 + \delta)\)-coreset of \( P \cup P' \), sends this coreset to \( \mathcal{A}_{j+1} \), and discards both \( P \) and \( P' \).

Thus, \( \mathcal{A}_0 \) uses space \( O(B) \) and, for each \( j \geq 1 \), \( \mathcal{A}_j \) uses space \( O(A(\delta)) \). The highest-numbered \( \mathcal{A}_j \) that we need is at level \( \lceil \log \left( \frac{m}{B} \right) \rceil \). This gives an overall space bound of \( O(B + A(\varepsilon/\log m)\lceil \log \left( \frac{m}{B} \right) \rceil) \), by our choice of \( \delta \).

Finally, by repeatedly applying the reduce property and the disjoint union property, we see that the final coreset, \( Q \), computed at the highest ("root") level is an \( \alpha \)-coreset, where

\[
\alpha = (1 + \delta)^{1 + \lceil \log \left( \frac{m}{B} \right) \rceil} \leq 1 + \delta \log m = 1 + \varepsilon.
\]

To estimate \( \inf_{x \in \mathbb{R}^d} C_\sigma(x) \), we simply output \( \inf_{x \in \mathbb{R}^d} C_Q(x) \), which we can compute directly.

As a corollary, we see that we can estimate the radius of the minimum enclosing ball (MEB), in \( d \) dimensions, up to a factor of \( 1 + \varepsilon \), by a data stream algorithm that uses space

\[
O \left( \frac{\log^{(d+1)/2} m}{\varepsilon^{(d-1)/2}} \right).
\]

In two dimensions, this amounts to \( O(\varepsilon^{-1/2} \log^{3/2} m) \).
Metric Streams and Clustering

In Lecture 12, we considered streams representing geometric data, and considered one class of computations on such data: namely, estimating extent measures. The measure we studied in detail — minimum enclosing ball (MEB) — can be thought of as follows. The center of the MEB is a crude *summary* of the data stream, and the radius of the MEB is the cost of thus summarizing the stream.

Often, our data is best summarized not by a single point, but by \( k \) points \((k \geq 1)\); one imagines that the data naturally falls into \( k \) clusters, each of which can be summarized by a representative point. For the problem we study today, these representatives will be required to come from the original data. In general, one can imagine relaxing this requirement. At any rate, a particular clustering has an associated *summarization cost* which should be small if the clusters have small extent (according to some extent measure) and large otherwise.

### 13.1 Metric Spaces

It turns out that clustering problems are best studied in a setting more general than the geometric one. The only aspect of geometry that matters for this problem is that we have a notion of “distance” between two points. This is abstracted out in the definition of a metric space, which we give below.

**Definition 13.1.1.** A metric space is a pair \((M,d)\), where \(M\) is a nonempty set (of “points”) and \(d : M \times M \to \mathbb{R}_+\) is a non-negative-valued “distance” function satisfying the following properties for all \(x,y,z \in M\).

1. \(d(x,y) = 0 \iff x = y\); \hspace{1cm} (identity)
2. \(d(x,y) = d(y,x)\); \hspace{1cm} (symmetry)
3. \(d(x,y) \leq d(x,z) + d(z,y)\). \hspace{1cm} (triangle inequality)

Relaxing the first property to \(d(x,x) = 0\) gives us a *semi-metric space* instead.

A familiar example of a metric space is \(\mathbb{R}^n\), under the distance function \(d(x,y) = \|x - y\|_p\), where \(p > 0\). In fact, the case \(p = 2\) (Euclidean distance) is especially familiar. Another example should be just about as familiar to computer scientists: take an (undirected) graph \(G = (V,E)\), let \(M = V\), and for \(u,v \in V\), define \(d(u,v)\) to be the length of the shortest path in \(G\) between \(u\) and \(v\).

At any rate, in this lecture, we shall think of the data stream as consisting of points in a metric space \((M,d)\). The function \(d\) is made available to us through an *oracle* which, when queried with two points \(x,y \in M\), returns the distance \(d(x,y)\) between them. To keep things simple, we will not bother with the issues of representing, in working memory, points in \(M\) or distance values. Instead we will measure our space usage as the number of points and/or distances our algorithms store.
13.2 The Cost of a Clustering: Summarization Costs

Fix a metric space \((M, d)\) and an integer \(k \geq 1\). Given a data set \(\sigma \subseteq M\), we wish to cluster it into at most \(k\) clusters, and summarize it by choosing a representative from each cluster. Suppose our set of chosen representatives is \(R\).

If we think of the elements of \(\sigma\) as being locations of “customers” seeking some service, and elements of \(R\) as locations of “service stations,” then one reasonable objective to minimize is the maximum distance that a customer has to travel to receive service. This is formalized as the \(k\)-center objective.

If we think of the elements of \(R\) as being locations of conference centers (for a multi-location video conference) and elements of \(\sigma\) as being home locations for the participants at this conference, another reasonable objective to minimize is the total fuel spent in getting all participants to the conference centers. This is formalized as the \(k\)-median objective. There is also another natural objective called \(k\)-means which is even older, is motivated by statistics and machine learning applications, and dates back to the 1960s.

To formally define these objectives, extend the function \(d\) by defining

\[ d(x, S) := \min_{y \in S} d(x, y), \]

for \(x \in M\) and \(S \subseteq M\). Then, having chosen representatives \(R\), the best way to cluster \(\sigma\) is to assign each point in \(\sigma\) to its nearest representative in \(R\). This then gives rise to (at least) the following three natural cost measures.

\[
\begin{align*}
\Delta_{\infty}(\sigma, R) &:= \max_{x \in \sigma} d(x, R); & (k\text{-center}) \\
\Delta_1(\sigma, R) &:= \sum_{x \in \sigma} d(x, R); & (k\text{-median}) \\
\Delta_2(\sigma, R) &:= \sum_{x \in \sigma} d(x, R)^2. & (k\text{-means})
\end{align*}
\]

Our goal is choose \(R \subseteq \sigma\) with \(|R| \leq k\) so as to minimize the cost \(\Delta(\sigma, R)\). We can build these restrictions on \(R\) into the definition of \(\Delta\) by setting \(\Delta(\sigma, R) = \infty\) whenever \(R\) does not meet the restrictions. For the rest of this lecture we focus on the first of these costs (i.e., the \(k\)-center problem).

We shall give an efficient data stream algorithm that reads \(\sigma\) as an input stream and produces a summary \(R\) whose cost is at most some constant \(\alpha\) times the cost of the best summary. Such an algorithm is called an \(\alpha\)-approximation. In fact, we shall give two such algorithms: the first will use just \(O(k)\) space and produce a 4-approximation. The second will improve the approximation ratio from 4 to \(2 + \varepsilon\), blowing up the space usage by about \(O(1/\varepsilon)\).

As noted earlier, when we produce a set of representatives, \(R\), we have in fact produced a clustering of the data implicitly: to form the clusters, simply assign each data point to its nearest representative, breaking ties arbitrarily.

13.3 The Doubling Algorithm

We focus on the \(k\)-center problem. The following algorithm maintains a set \(R\) consisting of at most \(k\) representatives from the input stream; these representatives will be our cluster centers. The algorithm also maintains a “threshold” \(\tau\) throughout; as we shall soon see, \(\tau\) approximates the summarization cost \(\Delta_{\infty}(\sigma, R)\), which is the cost of the implied clustering.

To analyze this algorithm, we first record a basic fact about metric spaces and the cost function \(\Delta_{\infty}\). Then we consider the algorithm’s workings and establish certain invariants that it maintains.

**Lemma 13.3.1.** Suppose \(x_1, \ldots, x_{k+1} \in \sigma \subseteq M\) satisfy \(d(x_i, x_j) \geq t\) for all distinct \(i, j \in [k+1]\). Then, for all \(R \subseteq M\), we have \(\Delta_{\infty}(\sigma, R) \geq t/2\).

**Proof.** Suppose there exists \(R \subseteq M\) with \(\Delta_{\infty}(\sigma, R) < t/2\). Then \(|R| \leq k\), and by the pigeonhole principle, there exist distinct \(i, j \in [k+1]\), such that \(\text{rep}(x_i, R) = \text{rep}(x_j, R) = r\), say. Now

\[
d(x_i, x_j) \leq d(x_i, r) + d(x_j, r) < t/2 + t/2 = t,
\]

\[57\]
where the first inequality is a triangle inequality and the second follows from $\Delta_w(\sigma, R) < t/2$. But this contradicts the given property of $\{x_1, \ldots, x_k+1\}$.

**Lemma 13.3.2.** The doubling algorithm maintains the following invariants at the start of each call to the processing section.

1. For all distinct $r, s \in R$, we have $d(r, s) \geq \tau$.

2. We have $\Delta_w(\sigma, R) \leq 2\tau$.

**Proof.** At the end of initialization, Invariant 1 holds by definition of $x, y$, and $\tau$. We also have $\Delta_w(\sigma, R) = \tau$ at this point, so Invariant 2 holds as well. Suppose the invariants hold after we have read an input stream $\sigma$, and are just about to process a new point $x$. Let us show that they continue to hold after processing $x$.

First, we consider the case that the if condition in Line 5 does not hold. Then $R$ and $\tau$ do not change, so Invariant 1 continues to hold. We do change $\sigma$ by adding $x$ to it. However, noting that $d(x, R) \leq 2\tau$, we have

$$\Delta_w(\sigma \cup \{x\}, R) = \max\{\Delta_w(\sigma, R), d(x, R)\} \leq 2\tau,$$

and so, Invariant 2 also continues to hold.

Next, suppose that the if condition in Line 5 holds. After Line 6 executes, Invariant 1 continues to hold because the point $x$ newly added to $R$ satisfies its conditions. And Invariant 2 continues to hold since $x$ is added to both $\sigma$ and $R$, which means $d(x, R) = 0$ and $d(y, R)$ does not increase for any $y \in \sigma \setminus \{x\}$; therefore $\Delta_w(\sigma, R)$ does not change.

We shall now show that the invariants are satisfied after each iteration of the while loop at Line 7. Invariant 1 may be broken by Line 8 but is explicitly restored in Line 9. Just after Line 8, with $\tau$ doubled, Invariant 2 is temporarily strengthened to $\Delta_w(\sigma, R) \leq \tau$. Now consider the set $R'$ computed in Line 9. To prove that Invariant 2 holds after that line, we need to prove that $\Delta_w(\sigma, R') \leq 2\tau$.

Let $x \in \sigma$ be an arbitrary data point. Then $d(x, R) \leq \Delta_w(\sigma, R) \leq \tau$. Let $r' = \arg\min_{r \in R} d(x, r)$. If $r' \in R'$, then $d(x, R') \leq d(x, r') = d(x, R) \leq \tau$. Otherwise, by maximality of $R'$, there exists a representative $s \in R'$ such that $d(r', s) < \tau$. Now

$$d(x, R') \leq d(x, s) \leq d(x, r') + d(r', s) < d(x, R) + \tau \leq 2\tau.$$

Thus, for all $x \in \sigma$, we have $d(x, R') \leq 2\tau$. Therefore, $\Delta_w(\sigma, R') \leq 2\tau$, as required.

Having established the above properties, it is now simple to analyze the doubling algorithm.

**Theorem 13.3.3.** The doubling algorithm uses $O(k)$ space and outputs a summary $R$ whose cost is at most 4 times the optimum.

**Proof.** The space bound is obvious. Let $R^*$ be an optimum summary of the input stream $\sigma$, i.e., one that minimizes $\Delta_w(\sigma, R^*)$. Let $\hat{R}$ and $\hat{T}$ be the final values of $R$ and $\tau$ after processing $\sigma$. By Lemma 13.3.2 (Invariant 2), we have $\Delta_w(\sigma, \hat{R}) \leq 2\hat{T}$.

By the same lemma (Invariant 1), every pair of distinct points in $\hat{R}$ is at distance at least $\hat{T}$. Therefore, by Lemma 13.3.1, we have $\Delta_w(\sigma, R^*) \geq \hat{T}/2$. Putting these together, we have $\Delta_w(\sigma, \hat{R}) \leq 4\Delta_w(\sigma, R^*)$. □

### 13.4 Metric Costs and Threshold Algorithms

The following two notions are key for our improvement to the above approximation factor.

**Definition 13.4.1.** A summarization cost function $\Delta$ is said to be **metric** if for all streams $\sigma, \pi$ and summaries $S, T$, we have

$$\Delta(\sigma[S] \circ \pi, T) - \Delta(\sigma, S) \leq \Delta(\sigma \circ \pi, T) \leq \Delta(\sigma[S] \circ \pi, T) + \Delta(\sigma, S). \quad (13.1)$$

Here, $\sigma[S]$ is the stream obtained by replacing each token of $\sigma$ with its best representative from $S$. 

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Importantly, if we define “best representative” to be the “nearest representative,” then the $k$-center cost function $\Delta$ is metric (an easy exercise). Also, because of the nature of the $k$-center cost function, we may as well replace $\sigma[S]$ by $S$ in (13.1).

Consider the following example of a stream, with ‘◦’ representing “normal” elements in the stream and ‘⊗’ representing elements that have been chosen as representatives. Suppose a clustering/summarization algorithm is running on this stream, has currently processed $\sigma$ and computed its summary $S$, and is about to process the rest of the stream, $\pi$.

The definition of a metric cost function attempts to control the “damage” that would be done if the algorithm were to forget everything about $\sigma$ at this point, except for the computed summary $S$. We think of $T$ as summarizing the whole stream, $\sigma \circ \pi$.

**Definition 13.4.2.** Let $\Delta$ be a summarization cost function and let $\alpha \geq 1$ be a real number. An $\alpha$-threshold algorithm for $\Delta$ is one that takes an input a threshold $t$ and a data stream $\sigma$, and does one of the following two things.

1. Produces a summary $S$; if so, we must have $\Delta(\sigma, S) \leq \alpha t$.
2. Fails (producing no output); if so, we must have $\forall T : \Delta(\sigma, T) > t$.

The doubling algorithm contains the following simple idea for a 2-threshold algorithm for the $k$-center cost $\Delta$. Maintain a set $S$ of representatives from $\sigma$ that are pairwise $2t$ apart; if at any point we have $|S| > k$, then fail; otherwise, output $S$. Lemma 13.3.1 guarantees that this is a 2-threshold algorithm.

### 13.5 Guha’s Algorithm

To describe Guha’s algorithm, we generalize the $k$-center problem as follows. Our task is to summarize an input stream $\sigma$, minimizing a summarization cost given by $\Delta$, which

- is a metric cost; and
- has an $\alpha$-approximate threshold algorithm $\mathcal{A}$, for some $\alpha \geq 1$.

As we have just seen, $k$-center has both these properties.

The idea behind Guha’s algorithm is to run multiple copies of $\mathcal{A}$ in parallel, with geometrically increasing thresholds. Occasionally a copy of $\mathcal{A}$ will fail; when it does, we start a new copy of $\mathcal{A}$ with a much higher threshold to take over from the failed copy, using the failed copy’s summary as its initial input stream.

Here is an outline of the algorithm, which computes an $(\alpha + O(\varepsilon))$-approximation of an optimal summary, for $\varepsilon \ll \alpha$. Let $S^*$ denote an optimal summary, i.e., one that minimizes $\Delta(\sigma, S^*)$. It should be easy to flesh this out into complete pseudocode; we leave this as an exercise.

1. Perform some initial processing to determine a lower bound, $c$, on $\Delta(\sigma, S^*)$.

2. Let $p = \lceil \log_{1+\varepsilon}(\alpha/\varepsilon) \rceil$. From now on, keep $p$ instances of $\mathcal{A}$ running at all times, with thresholds increasing geometrically by factors of $(1 + \varepsilon)$. The lowest threshold is initially set to $c(1 + \varepsilon)$.

3. Whenever $q \leq p$ of the instances fail, start up $q$ new instances of $\mathcal{A}$ using the summaries from the failed instances to “replay” the stream so far. When an instance fails, kill all other instances with a lower threshold. Alternatively, we can pretend that when an instance with threshold $c(1 + \varepsilon)^j$ fails, its threshold is raised to $c(1 + \varepsilon)^{j+p}$.

4. Having processed all of $\sigma$, output the summary from the instance of $\mathcal{A}$ that has the lowest threshold.
13.5.1 Space Bounds

In the above algorithm, let \( s_0 \) denote the space required to determine the initial lower bound, \( c \). Also, let \( s_{\text{inf}} \) denote the space required by an instance of \( \mathcal{A} \); we assume that this quantity is independent of the threshold with which \( \mathcal{A} \) is run. Then the space required by the above algorithm is

\[
\max\{s_0, ps_{\text{inf}}\} = O\left(s_0 + \frac{s_{\text{inf}}}{\epsilon} \log \frac{\alpha}{\epsilon}\right).
\]

In the case of \( k \)-center, using the initialization section of the Doubling Algorithm to determine \( c \) gives us \( s_0 = O(k) \). Furthermore, using the 2-threshold algorithm given at the end of Section 13.4, we get \( s_{\text{inf}} = O(k) \) and \( \alpha = 2 \). Therefore, for \( k \)-center, we have an algorithm running in space \( O((k/\epsilon) \log(1/\epsilon)) \).

13.5.2 The Quality of the Summary

Consider a run of Guha’s algorithm on an input stream \( \sigma \). Consider the instance of \( \mathcal{A} \) that had the smallest threshold (among the non-failed instances) when the input ended. Let \( t \) be the final threshold being used by this instance. Suppose this instance had its threshold raised \( j \) times overall. Let \( \sigma_i \) denote portion of the stream \( \sigma \) between the \((i-1)\)th and \(i\)th raising of the threshold, and let \( \sigma_{j+1} \) denote the portion after the last raising of the threshold. Then

\[
\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{j+1},
\]

Let \( S_i \) denote the summary computed by this instance of \( \mathcal{A} \) after processing \( \sigma_i \); then \( S_{j+1} \) is the final summary. During the processing of \( S_i \), the instance was using threshold \( t_i = t/(1+\epsilon)^{p(j-i+1)} \). Since \( p = \lceil \log_{1+\epsilon}(\alpha/\epsilon) \rceil \), we have \((1+\epsilon)^p \geq \alpha/\epsilon \), which gives \( t_j \leq (\epsilon/\alpha)^{j-i+1}t \). Now, by Property 1 of an \( \alpha \)-threshold algorithm, we have

\[
\Delta(S_{i-1} \circ \sigma_i, S_i) \leq \alpha(\epsilon/\alpha)^{j-i+1}t, \quad \text{for } 1 \leq i \leq j+1, \quad (13.2)
\]

where we put \( S_0 = \emptyset \). Since \( \Delta \) is metric, by (13.1), after the simplification \( \sigma[S] = S \), we have

\[
\Delta(\sigma_1 \circ \cdots \circ \sigma_i, S_i) \leq \Delta(S_{i-1} \circ \sigma_i, S_i) + \Delta(\sigma_1 \circ \cdots \circ \sigma_{i-1}, S_{i-1}). \quad (13.3)
\]

Using (13.3) repeatedly, we can bound the cost of the algorithm’s final summary as follows.

\[
\Delta(\sigma, S_{j+1}) = \Delta(\sigma_1 \circ \cdots \circ \sigma_{j+1}, S_{j+1}) \leq \sum_{i=1}^{j+1} \Delta(S_{i-1} \circ \sigma_i, S_i) \\
\leq \sum_{i=1}^{j+1} \alpha(\epsilon/\alpha)^{j-i+1}t \quad \text{(by (13.2))}
\]

\[
\leq \alpha \sum_{i=0}^{\infty} \left( \frac{\epsilon}{\alpha} \right)^i = (\alpha + O(\epsilon))t.
\]

Meanwhile, since \( t \) was the smallest threshold for a non-failed instance of \( \mathcal{A} \), we know that \( \mathcal{A} \) fails when run with threshold \( t/(1+\epsilon) \). By Property 2 of an \( \alpha \)-threshold algorithm, we have

\[
\Delta(\sigma, S^*) \geq \frac{t}{1+\epsilon}.
\]

Strictly speaking, the above reasoning assumed that at least one instance of \( \mathcal{A} \) failed while processing \( \sigma \). But notice that, by our choice of \( c \) in Guha’s algorithm, the above inequality holds even if this isn’t true, because in that case, we would have \( t = c(1+\epsilon) \).

Putting the last two inequalities together, we see that \( \Delta(\sigma, S_{j+1}) \) approximates the optimal cost \( \Delta(\sigma, S^*) \) within a factor of \((1+\epsilon)(\alpha + O(\epsilon)) = \alpha + O(\epsilon) \).

For \( k \)-center, since we have a 2-threshold algorithm, we get an overall approximation ratio of \( 2 + O(\epsilon) \).
Graph Streams: Basic Algorithms

14.1 Streams that Describe Graphs

We have been considering streams that describe data with some kind of structure, such as geometric structure, or more generally, metric structure. Another very important class of structured large data sets is large graphs. This motivates the study of graph algorithms that operate in streaming fashion: the input is a stream that describes a graph.

The model we shall work with in this course is that the input stream consists of tokens \((u, v) \in [n] \times [n]\), describing the edges of a simple\(^1\) graph \(G\) on vertex set \([n]\). We assume that each edge of \(G\) appears exactly once in the stream. There is no easy way to check that this holds, so we have to take this as a promise. The number \(n\) is known beforehand, but \(m\), the length of the stream and the number of edges in \(G\), is not. Though we can consider both directed and undirected graphs in this model, we shall only be studying problems on undirected graphs; so we may as well assume that the tokens describe doubleton sets \(\{u, v\}\).

Unfortunately, most of the interesting things we may want to compute for a graph provably \(\Omega(n)\) space in this model, even allowing multiple passes over the input stream. We shall show such results when we study lower bounds, later in the course. These include such basic questions as “Is \(G\) connected?” and even “Is there a path from \(u\) to \(v\) in \(G\)?” where the vertices \(u\) and \(v\) are known beforehand. Thus, we have to reset our goal. Where \((\log n)^{O(1)}\) space used to be the holy grail for basic data stream algorithms, for several graph problems, it is \(n(\log n)^{O(1)}\) space. Algorithms achieving such a space bound are sometimes called “semi-streaming” algorithms.\(^2\)

14.2 The Connectedness Problem

Our first problem is: decide whether or not the input graph \(G\), which is given by a stream of edges, is connected. This is a Boolean problem — the answer is either 0 (meaning “no”) or 1 (meaning “yes”) — and so we require an exact answer. We could consider randomized algorithms, but we won’t need to.

For this problem, as well as all others in this lecture, the algorithm will consist of maintaining a subgraph of \(G\) satisfying certain conditions. For connectedness, the idea is to maintain a spanning forest, \(F\), of \(G\). As \(G\) gets updated, \(F\) might or might not become a tree at some point. Clearly \(G\) is connected iff it does.

The algorithm below maintains \(F\) as a set of edges. The vertex set is always \([n]\).

We have already argued the algorithm’s correctness. Its space usage is easily seen to be \(O(n \log n)\), since we always have \(|F| \leq n - 1\), and each of \(F\) requires \(O(\log n)\) bits to describe.

The well known UNION-FIND data structure can be used to do the work in the processing section quickly. To

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\(^1\)A simple graph is one with no loops and no parallel edges.

\(^2\)The term does not really have a formal definition. Some authors would extend it to algorithms running in \(O(n^{3/2})\) space, say.
test acyclicity of \( F \cup \{u, v\} \), we simply check if root\( (u) \) and root\( (v) \) are distinct in the data structure. Note that this algorithm assumes an insertion-only graph stream: edges only arrive and never depart from the graph. All algorithms in this lecture will make this assumption.

### 14.3 The Bipartiteness Problem

A bipartite graph is one whose vertices can be partitioned into two disjoint sets, \( S \) and \( T \) say, so that every edge is between a vertex in \( S \) and a vertex in \( T \). Equivalently, a bipartite graph is one whose vertices can be properly colored using two colors.\(^3\) Our next problem is to determine whether the input graph \( G \) is bipartite.

Note that being bipartite is a monotone property (just as connectedness is): that is, given a non-bipartite graph, adding edges to it cannot make it bipartite. Therefore, once a streaming algorithm detects that the edges seen so far make the graph non-bipartite, it can stop doing more work. Here is our proposed algorithm.

Just like our connectedness algorithm before, this one also maintains the invariant that \( F \) is a subgraph of \( G \) and is a forest. Therefore it uses \( O(n \log n) \) space. Its correctness is guaranteed by the following theorem.

**Theorem 14.3.1.** The above algorithm outputs 1 iff the input graph \( G \) is bipartite.

**Proof.** Suppose the algorithm outputs 0. Then \( G \) must contain an odd cycle. This odd cycle does not have a proper 2-coloring, so neither does \( G \). Therefore \( G \) is not bipartite.

Next, suppose the algorithm outputs 1. Let \( \chi : [n] \to \{0, 1\} \) be a proper 2-coloring of the final forest \( F \) (such a \( \chi \) clearly exists). We claim that \( \chi \) is also a proper 2-coloring of \( G \), which would imply that \( G \) is bipartite and complete the proof.

To prove the claim, consider an edge \( e = \{u, v\} \) of \( G \). If \( e \in F \), then we already have \( \chi(u) \neq \chi(v) \). Otherwise, \( F \cup \{e\} \) must contain an even cycle. Let \( \pi \) be the path in \( F \) obtained by deleting \( e \) from this cycle. Then \( \pi \) runs between \( u \) and \( v \) and has odd length. Since every edge on \( \pi \) is properly colored by \( \chi \), we again get \( \chi(u) \neq \chi(v) \). \( \square \)

### 14.4 Shortest Paths and Distance Estimation via Spanners

Now consider the problem of estimating the distance in \( G \) between two vertices that are revealed after the input stream has been processed. That is, build a small data structure, in streaming fashion, that can be used to answer queries of the form “what is the distance between \( x \) and \( y \)”?

Define \( d_G(x, y) \) to be the distance in \( G \) between vertices \( x \) and \( y \):

\[
d_G(x, y) = \min \{ \text{length}(\pi) : \pi \text{ is a path in } G \text{ from } x \text{ to } y \},
\]

where the minimum of an empty set defaults to \( \infty \). The following algorithm computes an estimate \( \hat{d}(x, y) \) for the distance \( d_G(x, y) \). It maintains a suitable subgraph \( H \) of \( G \) which, as we shall see, satisfies the following property.

\[
\forall x, y \in [n] : d_G(x, y) \leq d_H(x, y) \leq t \cdot d_G(x, y), \tag{14.1}
\]

where \( t \geq 1 \) is an integer constant. A subgraph \( H \) satisfying (14.1) is called a \( t \)-spanner of \( G \). Note that the left inequality trivially holds for every subgraph \( H \) of \( G \).

We now show that the final graph \( H \) constructed by the algorithm is a \( t \)-spanner of \( G \). This implies that the estimate \( \hat{d}(x, y) \) is a \( t \)-approximation to the actual distance \( d_G(x, y) \): more precisely, it lies in the interval \( [d_G(x, y), t \cdot d_G(x, y)] \).

Pick any two distinct vertices \( x, y \in [n] \). We shall show that (14.1) holds. If \( d_G(x, y) = \infty \), then clearly \( d_H(x, y) = \infty \) as well, and we are done. Otherwise, let \( \pi \) be the shortest path in \( G \) from \( x \) to \( y \), and let \( x = v_0, v_1, v_2, \ldots, v_k = y \) be the vertices on \( \pi \), in order. Then \( d_G(x, y) = k \).

Pick an arbitrary \( i \in [k] \), and let \( e = \{v_{i-1}, v_i\} \). If \( e \in H \), then \( d_H(v_{i-1}, v_i) = 1 \). Otherwise, \( e \notin H \), which means that at the time when \( e \) appeared in the input stream, we had \( d_{H'}(v_{i-1}, v_i) \leq t \), where \( H' \) was the value of \( H \) at that time.

\(^3\)A coloring is proper if, for every edge \( e \), the endpoints of \( e \) receive distinct colors.
Since $H'$ is a subgraph of the final $H$, we have $d_H(v_{i-1}, v_i) \leq t$. Thus, in both cases, we have $d_H(v_{i-1}, v_i) = t$. By the triangle inequality, it now follows that

$$d_H(x, y) \leq \sum_{i=1}^{k} d_H(v_{i-1}, v_i) \leq tk = t \cdot d_G(x, y),$$

which completes the proof, and hence implies the quality guarantee for the algorithm that we claimed earlier.

How much space does the algorithm use? Clearly, the answer is $O(|H| \log n)$, for the final graph $H$ constructed by it.

To estimate $|H|$, we note that, by construction, the shortest cycle in $H$ has length at least $t+2$. We can then appeal to a result in extremal graph theory to upper bound $|H|$, the number of edges in $H$.

### 14.4.1 The Size of a Spanner: High-Girth Graphs

The girth $\gamma(G)$ of a graph $G$ is defined to be the length of its shortest cycle; we set $\gamma(G) = \infty$ if $G$ is acyclic. As noted above, the graph $H$ constructed by our algorithm has $\gamma(H) \geq t + 2$. The next theorem places an upper bound on the size of a graph with high girth (see the paper by Alon, Hoory and Linial [AHL02] and the references therein for more precise bounds).

**Theorem 14.4.1.** Let $n$ be sufficiently large. Suppose the graph $G$ has $n$ vertices, $m$ edges, and $\gamma(G) \geq k$, for an integer $k$. Then

$$m \leq n + n^{1+1/[\ell - \frac{1}{2}]}.$$

**Proof.** Let $d = 2m/n$ be the average degree of $G$. If $d \leq 3$, then $m \leq 3n/2$ and we are done. Otherwise, let $F$ be the subgraph of $G$ obtained by repeatedly deleting from $G$ all vertices of degree less than $d/2$. Then $F$ has minimum degree at least $d/2$, and $F$ is nonempty, because the total number of edges deleted is less than $n \cdot d/2 = m$.

Put $\ell = \lfloor \frac{k - 1}{2} \rfloor$. Clearly, $\gamma(F) \geq \gamma(G) \geq k$. Therefore, for any vertex $v$ of $F$, the ball in $F$ centered at $v$ and of radius $\ell$ is a tree (if not, $F$ would contain a cycle of length at most $2\ell \leq k - 1$). By the minimum degree property of $F$, when we root this tree at $v$, its branching factor is at least $d/2 - 1 \geq 1$. Therefore, the tree has at least $(d/2 - 1)^\ell$ vertices. It follows that

$$n \geq \left(\frac{d}{2} - 1\right)^\ell = \left(\frac{m}{n} - 1\right)^\ell,$$

which implies $m \leq n + n^{1+1/\ell}$, as required.

Using $\left\lfloor \frac{k - 1}{2} \right\rfloor \geq \frac{k - 2}{2}$, we can weaken the above bound to

$$m = O\left(n^{1+2/(k-2)}\right).$$

Plugging in $k = t + 2$, we see that the $t$-spanner $H$ constructed by our algorithm has $|H| = O(n^{1+2/t})$. Therefore, the space used by the algorithm is $O(n^{1+2/t} \log n)$. In particular, we can 3-approximate all distances in a graph by a streaming algorithm in space $O(n^{1/3})$. 

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15 Finding Maximum Matchings

[The rest is unchecked; use at your own risk.]

15.1 The Problems

The number of graph problems that one could try to solve is huge and we will only be able to cover a small fraction. In the previous lecture, we considered some very basic graph problems that we could solve easily in streaming fashion. Now we turn to maximum matching, which is a more sophisticated problem. It is well-studied in the classical theory of algorithm design, and Edmonds’s polynomial-time algorithm for the problem remains one of the greatest achievements in the field. For graphs described by streams, we cannot afford computations of the type performed by Edmonds’s algorithm. But it turns out that we can achieve low space (in the semi-streaming sense) if we settle for an approximation algorithm.

Before moving on to other topics, we shall consider the problem of counting (or estimating) the number of triangles in a graph. This number becomes meaningful if one thinks of the input graph as a social network, for instance. It turns out that this estimation problem is a rare example of a natural graph problem where a truly low-space (as opposed to semi-streaming) solution is possible. Moreover, a sketching algorithm is possible, which means that we can solve the problem even in a turnstile model.

15.2 Maximum Cardinality Matching

A matching is defined to be a graph where every vertex has degree at most 1. Having fixed a vertex set, one can think of a matching as a set of edges no two of which share a vertex. The maximum cardinality matching (or simply maximum matching) problem, abbreviated as MCM, is to find a largest sized subgraph $M$ of a given graph $G$ such that $M$ is a matching.

**Problem:** We are given a graph stream (stream of edges). We want to find a maximum matching.

There are two types of maximum matchings that we will consider:

- Maximum cardinality matching(MCM) : We want to maximize the number of edges in the matching.
- Maximum weight matching(MWM) : In this case, edges are assigned weights and we want to maximize the sum of weights of the edges in the matching.

“Semi-Streaming” model : aim for space $\tilde{O}(n)$ or thereabouts [we just want to do better than $O(n^2)$ space usage]

Both the algorithms for MCM and MWM have the following common characteristics:
UNIT 15. FINDING MAXIMUM MATCHINGS

- Approximation improves with more number of passes
- Maintain a matching (in memory) of the subgraph seen so far

**Input:** Stream of edges of a \( n \)-vertex graph

Without space restrictions: We can compute a MCM, starting with an empty set, and try to increase the size of matching: By finding an **augmenting path**.

**Definition 15.2.1.** Augmenting Path: Path whose edges are alternately in \( M \) and not in \( M \), beginning and ending in unmatched vertices.

**Theorem 15.2.2.** Structure theorem: If there is no augmenting path, then the matching is maximum.

So, when we can no longer find an augmenting path, the matching we have is an MCM by the above theorem.

**Streaming Algorithm for MCM:** Maintain a **maximal** matching (cannot add any more edges without violating the matching property). In more detail:

- **Initialize:** \( M \leftarrow \emptyset \)
- **Process** \((u, v)\): If \( M \cup \{(u, v)\} \) is a matching, \( M \leftarrow M \cup \{(u, v)\} \).
- **Output:** \(|M|\)

**Theorem 15.2.3.** Let \( t \) denote the output of the above algorithm. If \( t \) is the size of MCM of \( G \), then \( t/2 \leq i \leq t \).

**Proof.** Let \( M^* \) be a MCM and \( |M^*| = t \). Suppose \( |M| < t/2 \). Each edge in \( M \) “kills” (prevents from being added to \( M \)) at most two edges in \( M^* \). Therefore, there exists an unkillled edge in \( M^* \) that could have been added to \( M \). So, \( M \) is not maximal which is a contradiction.

**State of the art algorithm for MCM:** For any \( \varepsilon \), can find a matching \( M \) such that \( (1 - \varepsilon)t \leq |M| \leq t \), using constant (depends on \( \varepsilon \)) number of passes.

Outline: Find a matching, as above, in the first pass. Passes 2, 3, ... find a “short” augmenting path (depending on \( \varepsilon \)) and increase the size of the matching. Generalized version of the above structure theorem for matchings gives us the quality guarantee.

**MWM algorithm:**

- **Initialize:** \( M \leftarrow \emptyset \)
- **Process** \((u, v)\): If \( M \cup \{(u, v)\} \) is a matching, then \( M \leftarrow M \cup \{(u, v)\} \). Else, let \( C = \{\text{edges of } M \text{ conflicting with } (u, v)\} \) (note \( |C| = 1 \text{ or } 2 \)). If \( wt(u, v) > (1 + \alpha) wt(C) \), then \( M \leftarrow (M - C) \cup \{(u, v)\} \).
- **Output:** \( \hat{w} = wt(M) \)

Space usage of the above algorithm is \( O(n \log n) \).

**Theorem 15.2.4.** Let \( M^* \) be a MWM. Then, \( k \cdot wt(M^*) \leq \hat{w} \leq wt(M^*) \) where \( k \) is a constant.

With respect to the above algorithm, we say that edges are “born” when we add them to \( M \). They “die” when they are removed from \( M \). They “survive” if they are present in the final \( M \). Any edge that “dies” has a well defined “killer” (the edge whose inclusion resulted in its removal from \( M \)).

We can associate a (“Killing”) tree with each survivor where survivor is the root, and the edge(s) that may have been “killed” by the survivor (at most 2), are its child nodes. The subtrees under the child nodes are defined recursively.

**Note:** The above trees, so defined, are node disjoint (no edge belongs to more than one tree) as every edge has a unique “killer”, if any.

Let \( S = \{\text{survivors}\} \), \( T(S) = \bigcup_{e \in S} \{\text{edges in the Killing tree(e) not including } e\} \) (descendants of \( e \))

**Claim 1:** \( wt(T(S)) \leq wt(S)/\alpha \)
Proof. Consider one tree, rooted at \( e \in S \).

\[
wt(\text{level}_i, \text{descendants}) \leq \frac{wt(\text{level}_{i-1}, \text{descendants})}{1 + \alpha}
\]

Because, \( wt(\text{edge}) \geq (1 + \alpha)wt(\{\text{edges killed by it}\}) \)

\[
\Rightarrow wt(\text{level}_i, \text{descendants}) \leq \frac{wt(e)}{(1 + \alpha)^i} \\
wt(\text{descendants}) \leq wt(e) \left( \frac{1}{1 + \alpha} + \frac{1}{(1 + \alpha)^2} + \ldots \right) \\
= wt(e) \frac{1}{1 + \alpha} \frac{1}{1 - \frac{1}{1 + \alpha}} \\
= wt(e) \frac{1}{1 + \alpha - 1} \\
= \frac{wt(e)}{\alpha}
\]

\[
wt(T(S)) = \sum_{e \in S} wt(\text{descendants of } e) \leq \sum_{e \in S} \frac{wt(e)}{\alpha} = \frac{wt(S)}{\alpha}
\]

Claim 2: \( wt(M^*) \leq (1 + \alpha)(wt(T(S)) + 2 \cdot wt(S)) \)

Proof. Let \( e_1^*, e_2^*, \ldots \) be the edges in \( M^* \) in the stream order. We will prove the claim by using the following charging scheme:

- If \( e_j^* \) is born, then charge \( wt(e_j^*) \) to \( e_j^* \) which is in \( T(S) \cup S \)
- If \( e_j^* \) is not born, this is because of 1 or 2 conflicting edges
  - One conflicting edge, \( e \): Note: \( e \in S \cup T(S) \). Charge \( wt(e_j^*) \) to \( e \). Since \( e_j^* \) could not kill \( e \), \( wt(e_j^*) \leq (1 + \alpha)wt(e) \)
  - Two conflicting edges \( e_1, e_2 \): Note: \( e_1, e_2 \in T(S) \cup S \). Charge \( wt(e_j^*) \frac{wt(e_j^*)}{wt(e_j^*) + wt(e_j^*)} \) to \( e_j \) for \( j = 1, 2 \). Since \( e_j^* \) could not kill \( e_1, e_2 \), \( wt(e_j^*) \leq (1 + \alpha)(wt(e_1) + wt(e_2)) \). As before, we maintain the property that weight charged to an edge \( e \leq (1 + \alpha)wt(e) \).
- If an edge is killed by \( e' \), transfer charge from \( e \) to \( e' \). Note: \( wt(e) \leq wt(e') \), so \( e' \) can indeed absorb this charge.

Edges in \( S \) may have to absorb \( 2(1 + \alpha) \) times their weight (conflict two edges in \( M^* \), etc). This proves the claim. 

By claim 1&2,

\[
wt(M^*) \leq (1 + \alpha) \left( \frac{wt(S)}{\alpha} + 2 \cdot wt(S) \right) \\
= (1 + \alpha) \left( 1 + 2 \alpha \right) wt(S) \\
= \left( \frac{1}{1 + 3 + 2 \alpha} \right) wt(S)
\]

Best choice for \( \alpha \) (which minimizes the above expression) is \( 1/\sqrt{2} \). This gives

\[
\frac{wt(M^*)}{3 + 2\sqrt{2}} \leq \hat{w} \leq wt(M^*)
\]

Open question: Can we improve the above result (with a better constant)?
Counting Triangles

Given a graph stream, we want to estimate the number of triangles in the graph. Some known results about this problem:

- We can’t multiplicatively approximate the number of triangles in $o(n^2)$ space.
- We can approximate the number of triangles up to some additive error
- If we are given that the number of triangles $\geq t$, then we can multiplicatively approximate it.

Given an input stream of $m$ edges of a graph on $n$ vertices with at least $t$ triangles, we can compute $(\varepsilon, \delta)$-approximation using space $O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta} \cdot \frac{mn}{t})$ as follows:

1. Pick an edge $(u, v)$ uniformly at random from the stream
2. Pick a vertex $w$ uniformly at random from $V \setminus \{u, v\}$
3. If $(u, w)$ and $(v, w)$ appear after $(u, v)$ in the stream, then output $m(n - 2)$ else output 0.

It can easily be shown that the expectation of the output of the above algorithm is equal to the number of triangles. As before we run copies of the above algorithm in parallel and take the average of their output to be the answer. Using Chebyshev’s inequality, from the variance bound, we get the space usage to be $O(\frac{1}{\varepsilon^2} \log \frac{1}{\delta} \cdot \frac{mn}{t})$.

**Bar-Yossef, Kumar, Sivakumar [BKS02] algorithm:** Uses space $\tilde{O}(\frac{1}{\varepsilon^2} \log \frac{1}{\delta} \cdot (\frac{mn}{t})^2)$. Even though the space usage of this algorithm is more than the previous one, the advantage of this algorithm is it is a “sketching” algorithm (computes not exactly a linear transformation of the stream but we can compose “sketches” computed by the algorithm of any two streams and it can handle edge deletions).

**Algorithm:** Given actual stream of edges, pretend that we are seeing virtual stream of triples $\{u, v, w\}$ where $u, v, w \in V$. Specifically,

<table>
<thead>
<tr>
<th>actual token</th>
<th>Virtual token</th>
</tr>
</thead>
<tbody>
<tr>
<td>${u, v}$</td>
<td>${u, v, w_1}, {u, v, w_2}, \ldots, {u, v, w_{n-2}}$</td>
</tr>
</tbody>
</table>

where $\{w_1, w_2, \ldots, w_{n-2}\} = V \setminus \{u, v\}$.

Let $F_k$ be the $k^{th}$ frequency moment of virtual stream and

$$T_i = \left| \{u,v,w\} : u,v,w \text{ are distinct vertices and } \exists \text{ exactly } i \text{ edges amongst } u,v,w \right|$$
Note: $T_0 + T_1 + T_2 + T_3 = \binom{n}{3}$.

$$F_2 = \sum_{u,v,w} \text{(number of occurrences of } \{u,v,w\} \text{ in the virtual stream)}^2$$

$$= 1^2 \cdot T_1 + 2^2 \cdot T_2 + 3^2 \cdot T_3$$

$$= T_1 + 4T_2 + 9T_3$$

Similarly, $F_1 = T_1 + 2T_2 + 3T_3$ (Note: $F_1 = \text{length of virtual stream } = m(n - 2)$) and $F_0 = T_1 + T_2 + T_3$.

If we had estimates for $F_0, F_1, F_2$, we could compute $T_3$ by solving the above equations. So, we need to compute two sketches of the virtual stream, one to estimate $F_0$ and another to estimate $F_2$. 
Recall the Misra-Gries algorithm for MAJORITY problem where we found the majority element in a data stream in 2 passes. We can show that it is not possible to find the element using 1 pass in sublinear space. Let’s see how to determine such lower bounds.

### 17.1 Introduction to Communication Complexity

First let’s start with an introduction to an abstract model of communication complexity. There are two parties to this communication “game,” namely Alice and Bob. Alice’s input is \( x \in X \) and Bob’s input is \( y \in Y \). \( X \) and \( Y \) are known before the problem is specified. Now we want to compute \( f(x,y) : X \times Y \rightarrow Z \), when \( X = Y = [n] \), \( Z = \{0, 1\} \). For example,

\[
f(x,y) = (x+y) \mod 2 = x \mod 2 + y \mod 2
\]

In this case, Alice doesn’t have to send the whole input \( x \), using \( \lceil \log n \rceil \) bits; instead she can send \( x \mod 2 \) to Bob using just 1 bit! Bob calculates \( y \mod 2 \) and uses Alice’s message to find \( f(x,y) \). However, only Bob knows the answer. Bob can choose to send the result to Alice, but in this model, it is not required that all the players should know the answer. Note that we are not concerned about the memory usage, but we try to minimise the number of communication steps between Alice and Bob.

#### 17.1.1 EQUALITY problem

This problem finds its application in image comparisons. Given \( X = Y = \{0, 1\}^n \), \( Z = \{0, 1\} \)

\[
EQ(x,y) = \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise}
\end{cases}
\]

So in this case, communication between Alice and Bob requires \( n \) bits. For now, we consider a one-way transmission, ie messages are sent only in one direction. For symmetric functions, it doesn’t matter who sends the message to whom.

**Theorem 17.1.1.** Alice must send Bob \( n \) bits, in order to solve EQ in a one-way model.

\[
D^+(EQ) \geq n
\]
Proof: Suppose Alice sends $< n$ bits to Bob. Then the number of different messages she might send $\leq 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 2$. But Alice can have up to $2^n$ inputs.

Let’s recall the Pigeonhole principle. There are $n$ pigeons, $m$ holes and $m < n$. This implies that $\geq 2$ pigeons have to go into one hole.

Using the pigeonhole principle, there exists two different inputs $x \neq x'$, such that Alice sends the same message $\alpha$ on inputs $x$ and $x'$.

Let $P(x, y)$ be Bob’s output, when input is $(x, y)$. We should have

$$P(x, y) = EQ(x, y) \quad (17.1)$$

Since $P(x, y)$ is determined fully by Alice’s message on $x$ and Bob’s input $y$, using equation 13.1, we have

$$P(x, y) = EQ(x, x) = 1 \quad (17.2)$$

$$P(x', y) = EQ(x', x) = 0 \quad (17.3)$$

However since Bob sees the message $\alpha$ from Alice for both inputs $x$ and $x'$, $P(x, y) = P(x', y)$, which is a contradiction.

Hence the solution to the EQUALITY problem is for Alice to send all the $2^n$ messages using $n$ bits.

**Theorem 17.1.2.** Using randomness, we can compute $EQ$ function with an error probability $\leq 1/3$, in the one-way model, with message size $O(\log n)$

$$R^-(EQ) = O(\log n)$$

**Proof:** Consider the following protocol.

- Alice picks a random prime $p \in [n^2, 2n^2]$
- Alice sends Bob $(p, x \mod p)$, using $O(\log n)$ bits
- Bob checks if $y \mod p = x \mod p$, outputs 1 if true and 0 otherwise

If $EQ(x, y) = 1$, output is correct. If $EQ(x, y) = 0$, then its an error if and only if $(x-y) \mod p = 0$.

Remember that $x$ and $y$ are fixed and $p$ is random. Also we should choose a large prime for $p$, since if we choose $p = 2$, there is high error probability.

Let $x - y = p_1p_2\ldots p_t$ be the prime factorisation of $x - y$, where $p_1, p_2, \ldots, p_t$ need not be distinct. For the output $EQ(x, y)$ to be incorrect, $p \in \{p_1, p_2, \ldots, p_t\}$.

$$Pr[\text{error}] \leq \frac{t}{\text{no of primes in } [n^2, 2n^2]} \quad (17.4)$$

$$x - y \leq 2^n, \quad p_1p_2\ldots p_t \geq 2^t \Rightarrow t \leq n$$

Using prime number theorem,

$$\text{Number of primes in } [1..N] \approx \frac{N}{\ln N}$$

$$\text{Number of primes in } [n^2, 2n^2] \approx \frac{2n^2}{\ln 2n^2} - \frac{n^2}{\ln n^2}$$

$$= \frac{2n^2}{\ln 2 + 2 \ln n} - \frac{n^2}{2 \ln n}$$

$$\geq \frac{1.9 n^2}{2 \ln n} - \frac{n^2}{2 \ln n}$$

$$= \frac{0.9 n^2}{2 \ln n}$$

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Using estimates in equation 3, \( Pr[\text{error}] \) \leq \frac{n}{0.9 \, n^2 / 2 \ln n} = \frac{2 \ln n}{0.9 \, n} \leq \frac{1}{3}

### 17.2 Communication complexity in Streaming Algorithms

Now let's consider communication complexity models in streaming algorithms. Here the stream is cut into two parts, one part is with Alice and the other with Bob. We claim that if we can solve the underlying problem, we can solve the communication problem as well.

Suppose \( \exists \) a deterministic or randomised streaming algorithm to compute \( f(x, y) \) using \( s \) bits of memory, then

\[
D^\rightarrow(f) \leq s \quad \text{OR} \quad R^\rightarrow(f) \leq s
\]

Alice runs the algorithm on her part of the stream, sends the values in memory (\( s \) bits) to Bob, and he uses these values, along with his part of the stream, to compute the output.

**Note** one-pass streaming algorithm implies a one-way communication. So a

\[
p \text{ pass streaming algo } \Rightarrow \{ p \text{ messages from A } \rightarrow \text{ B} \}
\]

Hence we can generalize that a communication lower bound proven for a fairly abstract problem can be reduced to a streaming lower bound. i.e. from a streaming lower bound, we can reach the lower bound for a communication problem.

#### 17.2.1 INDEX problem

Given \( X = \{0, 1\}^n \), \( Y = [n] \), \( Z = \{0, 1\} \),

\[
\text{INDEX}(x, j) = x_j = j^{th} \text{ bit of } x
\]

e.g., \( \text{INDEX}(1100101, 3) = 0 \)

**Theorem 17.2.1.**

\[
D^\rightarrow(\text{INDEX}) \geq n
\]

**Proof** Can be proved using Pigeonhole Principle.

Say we have a MAJORITY streaming algorithm using \( s \) space. We are given an INDEX instance, Alice’s input : \( 1100101 \) and Bob’s input : \( 3 \). The scheme followed is

- ALICE’s input \( x \) is mapped to \( a_1a_2...a_m \), where \( a_i = 2(i-1) + x_i \)
- BOB’s input \( j \) is mapped to \( bb...b \), \( b \) occurs \( m \) times, where \( b = 2(j-1) \)
- Alice and Bob communicate by running \( A \) on \( \sigma, \pi \)
- If \( A \) says “no majority”, then output 1, else output 0.

1-pass MAJORITY algorithm requires \( \Omega(m) \) space. By theorem 13.2.1, communication uses \( \geq m, s \geq m = 1/2 \) stream length. We also have \( R^\rightarrow(\text{INDEX}) = \Omega(m) \).
Recap and an important point: Last time we saw that 1-pass MAJORITY requires space $\Omega(n)$ by reducing from INDEX. We say $\Omega(n)$, but in our streaming notation, $m$ is the stream length and $n$ is the size of the universe. We produced a stream $\sigma \circ \pi$, where $m = 2N$ and $n = 2N$, and we concluded that the space $s$ must be $\Omega(N)$. Looks like $s = \Omega(m)$ or $s = \Omega(n)$ but we have proven the lower bound only for the worst of the two. So, in fact, we have $s = \Omega(\min\{m,n\})$. For MAJORITY, we could say $\Omega(n)$ because $m$ and $n$ were about the same. We won’t be explicit about this point later, but most of our lower bounds have $\Omega(\min\{m,n\})$ form.

Today, we will see some communication lower bounds. But first, here is a small table of results:

<table>
<thead>
<tr>
<th>$f$</th>
<th>$D^{-}(f)$</th>
<th>$R_{1/3}^{-}(f)$</th>
<th>$D(f)$</th>
<th>$R_{1/3}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>INDEX</td>
<td>$\geq n$</td>
<td>$\Omega(n)$</td>
<td>$\leq \lceil \log n \rceil$</td>
<td>$\leq \lceil \log n \rceil$</td>
</tr>
<tr>
<td>EQUALITY</td>
<td>$\geq n$</td>
<td>$O(\log n)$</td>
<td>$\geq n$</td>
<td>$O(\log n)$</td>
</tr>
<tr>
<td>DISJOINTNESS</td>
<td>$\Omega(n)$</td>
<td>$\Omega(n)$</td>
<td>$\Omega(n)$</td>
<td>$\Omega(n)$</td>
</tr>
</tbody>
</table>

Table 18.1: In this table, we will either prove or have already seen almost all results except $R_{1/3}^{-}(EQ)$. Also, we will only prove a special case of DISJ.

We will start by proving $D(EQ) \geq n$. In order to prove this, however, we first have to prove an important property about deterministic communication protocols.

**Definition 18.0.1.** In two-way communication, suppose Alice first sends message $\alpha_1$ to Bob to which Bob responds with message $\beta_1$. Then, say Alice sends message $\alpha_2$ to Bob who responds with message $\beta_2$, and so on. Then, the sequence of messages $<\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots>$ is known as the *transcript* of the deterministic communication protocol.

**Theorem 18.0.2.** *Rectangle property of deterministic communication protocol:* Let $P$ be a deterministic communication protocol, and let $X$ and $Y$ be Alice’s and Bob’s input spaces. Let $\text{trans}_P(x,y)$ be the transcript of $P$ on input $(x,y) \in X \times Y$. Then, if $\text{trans}_P(x_1,y_1) = \text{trans}_P(x_2,y_2) = \tau$ (say); then $\text{trans}_P(x_1,y_2) = \text{trans}_P(x_2,y_1) = \tau$.

**Proof:** We will prove the result by induction.

Let $\tau = <\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots>$. Let $\tau' = <\alpha'_1,\beta'_1,\alpha'_2,\beta'_2,\ldots>$ be $\text{trans}_P(x_2,y_1)$.

Since Alice’s input does not change between $(x_2,y_1)$ and $(x_2,y_2)$, we have $\alpha'_1 = \alpha_1$.

Induction hypothesis: $\tau$ and $\tau'$ agree on the first $j$ messages.

**Case 1:** $j$ is odd $= 2k - 1$ ($k \geq 1$) (say).

So, we have: $\alpha'_1 = \alpha_1, \alpha'_2 = \alpha_2, \ldots, \alpha'_{k-1} = \alpha_{k-1}, \alpha'_k = \alpha_k$ and $\beta'_1 = \beta_1, \beta'_2 = \beta_2, \ldots, \beta'_{k-1} = \beta_{k-1}, \beta'_k = ?$.

Now, $\beta'_k$ depends on Bob’s input which does not change between $(x_1,y_1)$ and $(x_2,y_1)$, and it depends on the transcript so far which also does not change. So, $\beta'_k = \beta_k$. 72
**Case 2:** $j$ is even = $2k$ ($k \geq 1$) (say).

The proof for this case is similar to that of case 1, except we consider inputs $(x_2, y_2)$ and $(x_2, y_1)$.

By induction, we have the rectangle property.

Now that we have the rectangle property at hand, we can proceed to prove:

**Theorem 18.0.3.** $D(EQ) \geq n$.

**Proof:** Let $P$ be a deterministic communication protocol solving EQ. We will prove that $P$ has at least $2^n$ different transcripts which will imply that $P$ sends $\geq n$ bits.

Suppose $\text{trans}_{x}(x_1, x_1) = \text{trans}_{y}(x_2, x_2) = \tau$, where $x_1 \neq x_2$.

Then, by rectangle property:

\[
\text{trans}_{y}(x_2, x_2) = \tau \\
\Rightarrow \text{output on } (x_1, x_1) = \text{output on } (x_1, x_2) \\
\Rightarrow 1 = EQ(x_1, x_1) = EQ(x_1, x_2) = 0. \text{ [Contradiction]}
\]

This means that the $2^n$ possible inputs lead to $2^n$ distinct transcripts.

**Note:** General communication corresponds to multi-pass streaming.

**Theorem 18.0.4.** Any deterministic streaming algorithm for DISTINCT-ELEMENTS, i.e., $F_0$ must use space $\Omega(n/p)$, where $p$ is the number of passes.

Before seeing the proof, let us make note of a couple of points.

**Why $\Omega(n/p)$?** Over $p$ passes, Alice and Bob exchange $2p - 1$ messages; if the size of each message (i.e., space usage) is $s$ bits, then:

- total communication cost = $s(2p - 1)$
- total communication cost $\leq 2ps$.

If we can prove that the total communication must be at least $n$ bits, then we have:

- $n \leq 2ps$
- $s \geq n/2p$
- i.e., $s = \Omega(n/p)$

**Proof Idea:** We will reduce from EQ. Suppose Alice’s input $x \in \{0, 1\}^n$ is $\{100110\}$, and Bob’s input $y \in \{0, 1\}^n$ is $\{10110\}$. Using her input, Alice produces the stream $\sigma = \langle 1, 2, 4, 7, 9, 10 \rangle$ (i.e., $\sigma_i = 2(i - 1) + x_i$). Similarly, Bob produces the stream $\pi = \langle 1, 3, 4, 7, 9, 10 \rangle$.

Suppose Alice’s string disagrees in $d$ places with Bob’s string (i.e., Hamming distance $(x, y) = d$). Then, $F_0(\sigma \circ \pi) = n + d$.

If $\sigma$ and $\pi$ are equal, then $d = 0$, i.e., $EQ(x, y) = 1 \iff d = 0$.

So, $EQ(x, y) = 1 \implies d = 0 \implies F_0(\sigma \circ \pi) = n$

$EQ(x, y) = 0 \implies d \geq 1 \implies F_0(\sigma \circ \pi) \geq n + 1$

However, note that unless the $F_0$ algorithm is exact, it is very difficult to differentiate between $n$ and $n + (a \text{ small } d)$.

As we proved earlier, we can only get an approximate value for $F_0$, so we would like that if $x \neq y$, then Hamming distance between $x$ and $y$ be noticeably large. In order to get this large Hamming distance, we will first run Alice’s and Bob’s input through an encoder that guarantees a certain distance. For example, running Alice’s $n$-bit input $x$ through an encoder might lead to a $3n$-bit string $c(x)$ (similarly, $y$ and $c(y)$ for Bob), and the encoder might guarantee that $x \neq y \Rightarrow$ Hamming distance between $c(x)$ and $c(y) \geq 3n/100$ (say).

**Proof:** Reduction from EQ.

Suppose Alice and Bob have inputs $x, y \in \{0, 1\}^n$. Alice computes $\bar{x} = C(x)$, where $C$ is an error-correcting code with encoded length $3n$, and distance $\geq 3n/100$. Then, she produces a stream $\bar{\sigma} = \langle \sigma_1, \sigma_2, \ldots, \sigma_{3n} \rangle$, where $\sigma_i = 2(i - 1) + \bar{x}_i$.

Similarly, Bob has $y$, he computes $\bar{y} = C(y)$ and produces a stream $\bar{\pi}$ where $\pi_i = 2(i - 1) + \bar{y}_i$.

Now, they simulate $F_0$ streaming algorithm to estimate $F_0(\sigma \circ \pi)$.

If $x = y$, $EQ(x, y) = 1$, then $\bar{x} = \bar{y}$, so $F_0 = 3n$.

If $x \neq y$, $EQ(x, y) = 0$, then $\text{Hamming}(\bar{x}, \bar{y}) \geq 3n/100$, so $F_0 \geq (3n + 3n/100)$. That is $F_0 \geq 3n \times 1.01$. 

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If $F_0$ algorithm gives a $(1 \pm \varepsilon)$-approximation with $\varepsilon < 0.005$, then we can distinguish between the above two situations and hence solve EQ. But $D(EQ) \geq n$, therefore, $D(F_0) = \Omega(n/p)$ with $p$ passes.
Set Disjointness and Multi-Pass Lower Bounds

19.1 Communication Complexity of DISJ

Alice has a n-bit string x and Bob has a n-bit string y, where x, y ∈ [0, 1]n. x, y represent subset X, Y of [n] respectively. x, y are named characteristic vector or bitmask, i.e.

\[ x_j = \begin{cases} 1 & \text{if } j \in X \\ 0 & \text{otherwise} \end{cases} \]

Compute function

\[ \text{DISJ}(x, y) = \begin{cases} 1 & \text{if } X \cap Y \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \]

Theorem 19.1.1. \( R(\text{DISJ}) = \Omega(n) \)

Proof. We prove the special case: \( R^{-}(\text{DISJ}) = \Omega(n) \) by reduction from INDEX. Given an instance \((x, j)\) of INDEX, we create an instance \((x', y')\) of DISJ, where \(x' = x, y'\) is a vector s.t.

\[ y'_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otherwise} \end{cases} \]

By construction, \(x' \cap y' \neq \emptyset \iff x_j = 1\) i.e. \(\text{DISJ}(x', y') = \text{INDEX}(x', y')\). Thus, a one way protocol of DISJ implies a one way protocol of INDEX with the same communication cost. But \(R^{-}(\text{INDEX}) = \Omega(n)\), so we have \(R^{-}(\text{DISJ}) = \Omega(n)\).

The general case is proved by [Kalya], [Razborov '90] [Bar Yossof-Jayarom Kauor '02].

19.2 ST-Connectivity

ST-connectivity: Given an input graph stream, are two specific vertices s and t connected?

Theorem 19.2.1. Solving ST-CONN requires \( \Omega(n) \) space. (semi-streaming is necessary.)

Proof. By reduction from DISJ.
Let \((x, y)\) be an instance of DISJ. Construct an instance of CONN on a graph with vertex set \(\{s, t, v_1, v_2, \ldots, v_{n-2}\}\). So
Alice gets set of edges $A = \{(s, v) : v \in x\}$ and Bob gets set of edges $B = \{(v, t) : v \in y\}$.

Vertex $s$ and $t$ are connected in the resulting graph
\[\iff \exists \text{ a length-two path from } s \text{ to } t\]
\[\iff \exists v \text{ s.t. } (s, v) \in A \text{ and } (v, t) \in B\]
\[\iff \exists v \in x \cap y\]
\[\iff \text{DISJ}(x, y) = 1\]

Reducing this communication problem to a streaming problem, we have the required space in a $p$ pass algorithm is $\Omega(n/p)$.

\[\square\]

19.3 Perfect Matching Problem

Given a graph on $n$ vertices, does it have a perfect matching i.e. a matching of size $n/2$?

**Theorem 19.3.1.** One pass streaming algorithm for this problem requires $\Omega(n^2)$ space.

**Proof.** Reduction from INDEX.

Suppose we have an instance $(x, k)$ of INDEX where $x \in \{0, 1\}^N, k \in [N^2]$. Construct graph $G$, where $V_G = \bigcup_{i=1}^{N} \{a_i, b_i, u_i, v_i\}$ and $E_G = E_A \cup E_B$. Edges in $E_A$ and $E_B$ are introduced by Alice and Bob respectively.

\[E_A = \{(u_i, v_j) : x_{f(i, j)} = 1\}, \text{ where } f \text{ is a 1-1 correspondence between } [N] \times [N] \rightarrow [N^2]\]

E.g. $f(i, j) = (N - 1)i + j$

\[E_B = \{(a_i, u_j : l \neq i)\} \cup \{(b_i, v_j : l \neq j)\} \cup \{(a_i, b_j)\}, \text{ where } i, j \in [N] \text{ are s.t. } f(i, j) = k\]

By construction,

$G$ has a perfect matching
\[\iff (u_i, v_j) \in E_G\]
\[\iff x_k = 1\]
\[\iff \text{INDEX}(x, k) = 1\]

Thus, the space usage of a streaming algorithm for perfect matching must be $\Omega(N^2)$. The number of vertices in instance constructed is $n = 4N$. So the lower bound in term of $n$ is $\Omega((\frac{n}{2})^2) = \Omega(n^2)$.

\[\square\]

19.4 Multiparty Set Disjointness (DISJ$_{n,t}$)

Suppose there are $t$ player and each player $A_i$ has a $n$-bit string $x_i \in \{0, 1\}^n$. Define

\[
\text{DISJ}_{n,t} = \begin{cases} 
1 & \text{if } \bigcup_{i=1}^{t} x_i \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

**Theorem 19.4.1** (C.-Khot-Sun '03/Granmeier '08). $R(\text{DISJ}_{n,t}) = \Omega(n/t)$

**Theorem 19.4.2.** Approximating 1-pass $F_i$ with randomization allowed requires $\Omega(m^{1-2/k})$ space.

**Proof.** Given a DISJ instance $(x_1, x_2, \ldots, x_t)$ s.t. every column of matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{pmatrix}$ must contain 0,1 or $t$ ones and at most one column can have $t$ ones. Player $A_i$ converts $x_i$ into a stream $\pi_i = <\sigma_1, \sigma_2, \cdots >$ such that $\sigma_j \in \pi_i$ iff. $x_{ij} = 1$. 
Simulate the $F_k$ algorithm on stream $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_t$. The frequency vector of this stream is either 0, 1 only (if $\text{DISJ}_{n,t} = 0$) or 0, 1 and a single $t$ (if $\text{DISJ}_{n,t} = 1$). In the first case, $F_k \leq n$ while $F_k \geq t^k$ in the second case. Choose $t$ s.t. $t^k > 2m$, then 2-approximation to $F_k$ can distinguish these two situations. Thus, the memory usage is $\Omega(m/t^2) = \Omega(m^{1-2/k})$, for stream length $m \leq nt$ and $m = \Omega(1/t)$. \qed
Bibliography


